THE PRINCIPLES OF COMPOUND INTEREST

THE PRINCIPLES OF COMPOUND INTEREST

IN THEIR PRACTICAL APPLICATION TO ANNUITIES, REDEEMABLE SECURITIES SINKING FUNDS, LOAN TRANSACTIONS ETC.

BY

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LONDON
SIR ISAAC PITMAN & SONS, LTD.
PARKER STREET, KINGSWAY, W.C.2
BATH, MELBOURNE, TORONTO, NEW YORK
1925

PRINTED IN GREAT BRITAIN
AT THE PITMAN PRESS, BATH

PREFACE

THIS book has been written with the object of serving as a textbook for the subject of Compound Interest in the Chartered Insurance Institute Examination. In the past, this subject has presented very considerable difficulties to students preparing for this particular examination, and it is hoped that at least some of these difficulties will be smoothed away by the present book, which, it may be added, treats the subject from a slightly different standpoint from that which is usually taken up, in that the exposition is mainly arithmetical in character, instead of algebraical.

It is not possible, however, to deal with the subject exclusively from the arithmetical point of view, and the reader should therefore make sure that he can rely upon his algebra up to, say, Quadratic Equations, before he commences systematic study of the book. This point is of great importance, as an inadequate mathematical knowledge is usually to be found at the root of a student's difficulties. Two further matters call for attention here as regards the mathematical equipment of the reader on the arithmetical side. In the first place, facility in converting shillings and pence into their decimal equivalents of a £, and vice versa, is a great advantage; in the second place, the ready use of contracted methods of multiplication and division, such as are demonstrated in any modern textbook of Arithmetic, is practically an essential for an examination candidate, to whom the saving of time is naturally a most important consideration. Some portions of the book make it desirable that the reader should have a working knowledge of Logarithms, but those portions may be omitted if the reader requires merely an insight into the practical use of Compound Interest Tables.

Throughout the book, the principles expounded in the text have been fully illustrated by arithmetical examples, and a few exercises, to which the answers are given at the end of the book, have been added to each chapter, with a view to enabling the reader to obtain practical experience.

In conclusion, the author wishes to express his thanks to his colleague, Mr. R. Murrell, F.I.A., for reading the manuscript and for making several valuable criticisms and suggestions.

H. H. E.

CONTENTS

THE PRINCIPLES OF COMPOUND INTEREST

CHAPTER I

AMOUNTS AND PRESENT VALUES OF SINGLE SUMS

1. THE theory of compound interest assumes that the interest which is derived from the investment of capital is itself employed as capital, and productively invested accordingly, as soon as it is received.

If, for example, £100 were invested at a rate of interest of 5% per annum, the interest of £5 received at the end of the first year would then become additional capital available for investment, with the result that the total capital standing invested at the beginning of the second year would be £105. This increased capital of £105 would earn interest of £5 5s. during the second year, the capital being thereby further increased to £110 5s. at the end of that year, and so on from year to year.

2. The capital originally invested is usually referred to as the principal, and the total accumulation of capital at the end of a given period is known as the amount. The difference between the amount and the principal represents the compound interest which has been earned during the period. For instance, in the case of the above example, the compound interest earned during the two years is the difference between £110 5s. (the amount) and £100 (the principal), viz. £10 5s.

3. The accumulated amount of a given principal at the end of any prescribed number of years may always be found by proceeding on the lines of paragraph 1, although calculation by such a detailed method is never necessary in practice. It may, however, be useful to give a further example of the method, showing how the working may be conveniently set out; for this purpose it will be assumed that it is desired to find the compound interest on £135 for 5 years at a rate of

interest of 4% per annum. The working would be as follows—

Principal	£135
1st year's interest	5·4
Amount at end of 1st year . . .	140·4
2nd year's interest	5·616
Amount at end of 2nd year . .	146·016
3rd year's interest	5·8406
Amount at end of 3rd year . . .	151·8566
4th year's interest	6·0743
Amount at end of 4th year . . .	157·9309
5th year's interest	6·3172
Amount at end of 5th year . . .	164·2481
Deduct original principal . . .	135·0000
Compound interest for the 5 years .	£29·2481

or £29 5s. to the nearest penny.

4. It is evident that, if a long period were involved, the arithmetical work in a calculation of this kind would be considerable, and, for this reason alone, it is desirable to investigate better methods. It must first be explained that, whilst interest is customarily described as being at a certain rate *per cent* per annum, it is more convenient in developing the theory of compound interest to regard it as being at a certain rate per annum for each *unit* of capital invested. Thus, 5% per annum would be regarded as ·05 per unit per annum, $3\frac{3}{4}$% per annum as ·0375 per unit per annum, and so on.

5. If a unit of principal were invested at a rate of interest of i per unit per annum, the amount at the end of the first year would be $(1 + i)$, or $(1 + i)$ times the original unit. This amount of $(1 + i)$, being the capital at the beginning of the second year, would accumulate to $(1 + i)$ times itself, viz. to $(1 + i)^2$, by the end of the second year. By the end of the third year this capital of $(1 + i)^2$ would accumulate to $(1 + i)$ times itself, viz. to $(1 + i)^3$, and by continuing the reasoning it may be shown that the amount of the original principal at the end of any number of years, say n, would be $(1 + i)^n$.

In case this reasoning should not appear conclusive, an

alternative method of procedure is given which closely follows the arithmetical method exemplified in paragraph 3—

Principal	1
1st year's interest	
Amount at end of 1st year . .	$(1 + i)$
2nd year's interest . .	$i\,(1 + i)$
Amount at end of 2nd year . .	$(1 + i)^2$ *
3rd year's interest	$i\,(1 + i)^2$
Amount at end of 3rd year . . .	$(1 + i)^3$ †
And so on.	

It will be seen that the amount at the end of—

$$\begin{aligned} 1 \text{ year} &= (1 + i) \\ 2 \text{ years} &= (1 + i)^2 \\ 3 \text{ years} &= (1 + i)^3 \end{aligned}$$

And, by proceeding in this manner, it may be shown that the amount at the end of n years $= (1 + i)^n$ as before.

6. Since each unit of principal invested would accumulate to $(1 + i)^n$ by the end of n years, a principal of P would accumulate to an amount of $P\,(1 + i)^n$ by the end of the same period. This relationship between the principal and the amount at the end of n years is embodied in the formula—

$$S = P\,(1 + i)^n \quad . \quad . \quad . \quad (1)$$

where S is used to denote the amount.

As an illustration of the application of the formula, take the example of paragraph 3. Here $P = £135$; $i = \cdot 04$; and $n = 5$. So that—

$$\begin{aligned} S &= £135\,(1\cdot 04)^5 \\ &= £135 \times 1\cdot 21665 \\ &= £164\cdot 248, \text{ or } £164\ 5s. \text{ as before.} \end{aligned}$$

7. If necessary, the value of $(1 + i)^n$ could be found in each instance by means of logarithms, but this course is seldom adopted in practice, as the values of $(1 + i)^n$ are tabulated in compound interest tables at the more usual rates of interest for terms up to 50 or possibly 100 years. Even if the term should be beyond the range of the particular tables in use, the desired value could still be obtained from the tabulated

* Since $(1 + i) + i\,(1 + i)$ factorizes into $(1 + i)\,(1 + i) = (1 + i)^2$

† Since $(1 + i)^2 + i\,(1 + i)^2$ „ „ $(1 + i)^2\,(1 + i) = (1 + i)^3$

values by methods which will be explained later. Should, however, the rate of interest not be given in the tables, logarithms would have to be used, and formula (1) would accordingly be adapted as follows—

$$\log S = \log P + n \log (1 + i) \quad . \quad . \quad (2)$$

If, for example, it were desired to find the amount of £57 4s. 2d. accumulated for 15 years at a rate of interest of 5·4% per annum, the calculation by means of logarithms would be—

$$\begin{aligned} \log S &= \log 57{\cdot}208 + 15 \times \log 1{\cdot}054 \\ &= 1{\cdot}7574568 + 15 \times {\cdot}0228406 \\ &= 2{\cdot}1000658 \\ &= \log 125{\cdot}912. \end{aligned}$$

Whence the desired amount is £125 18s. 3d.

8. It has been seen how a sum of unity will accumulate to $(1 + i)^n$ in n years under the operation of compound interest at the rate of i per unit per annum, and the principal of 1 may, on the same basis as to interest, be regarded as the present value of a sum of $(1 + i)^n$ due at the end of n years. The present value of a sum due at a future date will thus accumulate to that sum by the time that date is reached. Hitherto the case has been considered where the principal (P), the rate of interest per unit per annum (i), and the term of years (n) have been given, and it has been required to find the amount (S). In the cognate problem of finding the present value, S, i, and n are given, and it is desired to find P. This may be done by simple proportion as follows—

The present value of $(1 + i)^n$ due at the end of n years is 1

Therefore the present value of 1 due at the end of n years is . . . . . . $\frac{1}{(1 + i)^n}$

And the present value of S due at the end of n years is . . . . . . $\frac{S}{(1 + i)^n}$

This result could have been obtained direct from formula (1) by dividing both sides by $(1 + i)^n$, whence

It is convenient, for certain reasons which will appear later, to denote $\frac{1}{(1+i)}$ by v. In this short notation, $\frac{1}{(1+i)^n}$ becomes v^n, and the last formula may be written

$$P = S.v^n \quad . \quad . \quad . \quad . \quad (3b)$$

9. To find the present value of a sum due at a future date, the sum in question has thus to be multiplied by the appropriate value of v^n. The values of v^n are tabulated in compound interest tables at all the usual rates of interest over a considerable range of years. If, however, for any reason, logarithms had to be employed, formula (3*a*) would be used rather than (3*b*).

Formula (3*a*) adapted to the use of logarithms is

$$\log P = \log S - n \log (1 + i) \quad . \quad . \quad . \quad . \quad (4)$$

For example, the present value at a rate of interest of 3·8% per annum of a sum of £89 4s. 2d. due 25 years hence would be calculated as follows—

$$\begin{aligned} \log P &= \log 89{\cdot}208 - 25 \times \log 1{\cdot}038 \\ &= 1{\cdot}9504038 - 25 \times {\cdot}0161974 \\ &= 1{\cdot}5454688 \\ &= \log 35{\cdot}113 \end{aligned}$$

The desired present value is therefore £35 2s. 3d.

Nominal Rates of Interest

10. It has been taken for granted so far that interest is payable yearly, but in practice it is quite common for rates of interest to be payable half-yearly, quarterly, or even monthly. When a rate of interest is payable more frequently than yearly, it is said to be at a *nominal* rate. In such cases, interest is still described as being at a certain rate per cent (or per unit) *per annum*, but the qualification is made that it is " payable half-yearly, quarterly, etc.," as the case might be. Sometimes the periodicity of payment is indicated by stating that interest is " convertible half-yearly, quarterly, etc.," or that it is " payable with half-yearly, quarterly, etc., rests." The symbol $j_{(m)}$ is employed in theoretical work to denote a nominal rate of interest of j per unit per annum payable m times a year: this symbol is often abbreviated to j simply, if the frequency of conversion, m, is clearly to be understood from the context.

When interest is payable *annually* at a certain rate it is said to be at an *effective* rate, and is usually denoted symbolically by i, as has already been seen.

In making calculations at a nominal rate of interest of $j_{(m)}$ it is necessary to regard the interest as being at an effective rate of $\frac{j}{m}$ during each interval of $\frac{1}{m}$th of a year, and to consider the total number of such *intervals* over which it is to operate rather than the number of *years*, as in the case of an effective rate. The unit of time-measurement is thus changed from one year to one interval. It may here be mentioned that it is always best to consider interest tables as showing factors appropriate not merely to certain rates of interest *per annum* and periods of *years*, but to certain rates of interest *per interval* and periods of *multiples of that interval*.

Suppose, for example, that it is desired to find the amount of a certain sum accumulated for 5 years at a nominal rate of interest of 4% per annum payable half-yearly: the rate of interest should be regarded as 2% per interval, and the period as 10 intervals. The factor by which the principal should be multiplied is therefore $(1{\cdot}02)^{10}$, the value of which would be given in the tables. The same principle would obviously apply with regard to the calculation of present values at nominal rates of interest.

11. In the case of a nominal rate of interest of $j_{(m)}$ the formulas corresponding to (1), (2), (3*a*), (3*b*), and (4) are therefore as follows—

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\log S = \log P + mn \log\left(1 + \frac{j}{m}\right) \quad . \quad . \quad . \quad (6)$$

$$P = S\, v^{mn} \text{ where } v = \frac{1}{\left(1 + \frac{j}{m}\right)} \quad . \quad . \quad . \quad (8)$$

$$\text{and } \log P = \log S - mn \log\left(1 + \frac{j}{m}\right) \quad (9)$$

12. It is sometimes required to find what effective rate of interest corresponds to a given nominal rate of interest payable with a given frequency, and vice versa. The effective rate of interest per unit per annum corresponding to a given nominal rate is the total compound interest which would be earned in one year from the investment of a unit of principal at the given nominal rate. If the nominal rate of interest per unit per annum were j, convertible m times a year, the total compound interest which would be earned from the investment of a unit of capital for one year would be $\left(1+\frac{j}{m}\right)^m - 1$, and this would give the desired equivalent effective rate of interest.

Thus, $$i = \left(1+\frac{j}{m}\right)^m - 1 \quad . \quad . \quad . \quad (10)$$

For example, the effective rate of interest per unit per annum which is equivalent to a nominal rate of 4% per annum payable quarterly would be given by $(1{\cdot}01)^4 - 1$, which equals ·04060, approximately.

13. To find the nominal rate convertible with any desired frequency corresponding to a given effective rate, it is necessary to use a modified formula, derived as follows—

From formula (10) $$\left(1+\frac{j}{m}\right)^m = (1+i)$$

Taking the m^{th} root of both sides of this equation,

1

Whence $$\frac{j}{m} = (1 +$$

And $$j = m\left\{(1+i)^{\frac{1}{m}} - 1\right\} \quad . \quad . \quad .$$

For example, given $i = {\cdot}04$, the corresponding nominal rate of interest convertible half-yearly would be given by

$$\begin{aligned} &2\left\{(1{\cdot}04)^{\frac{1}{2}} - 1\right\} \\ &= 2(1{\cdot}019804 - 1) \\ &= {\cdot}039608, \end{aligned}$$

the square root of (1·04) being found by logarithms, as follows—

$$\log 1{\cdot}04 = {\cdot}0170333$$

$$\begin{aligned} \text{And } \log(1{\cdot}04)^{\frac{1}{2}} &= \tfrac{1}{2}\log 1{\cdot}04 \\ &= {\cdot}0085167 \\ &= \log 1{\cdot}019804 \end{aligned}$$

14. If the nominal rate of interest is fixed, the more frequently it is convertible the greater becomes the equivalent effective rate. There is, however, a limit beyond which the latter cannot increase, no matter how frequently the nominal rate is convertible, and it may be shown that the value of this limit is approximately equal to $j + \frac{j^2}{2}$. For instance, the effective rate of interest (expressed to 4 places of decimals) corresponding to a nominal rate of ·04 per unit per annum can never exceed $\cdot 04 + \frac{(\cdot 04)^2}{2}$, or ·0408, even if the nominal rate were convertible momently.

Similarly, if the effective rate of interest is fixed, the corresponding nominal rate will decrease the more frequently it is convertible, but it cannot fall below a certain limit. If the effective rate of interest per unit per annum were i, this limit would be approximately $i - \frac{i^2}{2}$. If, for example, the effective rate of interest per unit per annum were ·05, the corresponding nominal rate could never be less than

[illegible] 2, [illegible]

Discount

15. The difference between a sum due at a future date and the present value, calculated at a specified rate of interest, is often referred to as the " Discount " on the sum in question. Thus, the discount on a sum of S due n years hence would be $S - Sv^n$, where v^n would be calculated at the predetermined rate of interest. For instance, to revert to the example in paragraph 9, the discount, calculated at a rate of interest of 3·8% per annum, on a sum of £89 4s. 2d. due 25 years hence would be £54 1s. 11d.

In commercial transactions, particularly where the period involved is short (as in the case of bill-discounting), it is customary to calculate discount as the simple interest on the amount of the debt for the period which is to elapse before it becomes due. It is the custom of bankers to calculate discount in this way, with the consequence that discount determined by this method has come to be known as "Bankers' Discount." For example, the bankers' discount, calculated at

a rate of $4\frac{1}{2}\%$ per annum on a sum of £135 18s. 7d. due 137 days hence would be $£135{\cdot}929 \times {\cdot}045 \times \frac{137}{365} = £2{\cdot}296 = £2$ 5s. 11d.

It should be added that it is usual in practice, although not rigidly accurate, to calculate interest and discount for fractions of an interval on the assumption of simple interest instead of compound interest. The following examples will suffice as illustrations in this respect. Suppose that it is desired to find the amount of a sum of £247 3s. 6d. accumulated for 9 years 118 days at a nominal rate of interest of 6% per annum, convertible quarterly. The total period involved is 9·3233 years, or 37·2932 quarters. The amount at the end of 37 quarters would be $£247{\cdot}175 \times (1{\cdot}015)^{37} = £247{\cdot}175 \times 1{\cdot}73478 = £428{\cdot}794$.

Now the simple interest on this amount for the final period of ·2932 of a quarter is $£428{\cdot}794 \times {\cdot}015 \times {\cdot}2932 = £1{\cdot}886$. So that the amount at the end of the total period of 37·2932 quarters is $£428{\cdot}794 + £1{\cdot}886 = £430{\cdot}680$ or £430 13s. 7d.

Suppose, again, that it is desired to find the present value, at a nominal rate of interest of 6% per annum convertible half-yearly, of a sum of £125 8s. 2d. due 4 years 205 days hence. The total period involved is 4·5616 years, or 9·1232 half-years. As at a point of time ·1232 of a half-year hence, the "present" value of the sum of £125 8s. 2d. would be $£125{\cdot}408 \times v^9$ calculated at 3%, or $£125{\cdot}408 \times {\cdot}76642 = £96{\cdot}115$.

The discount for the odd period of ·1232 of a half-year on this sum of £96·115 would be calculated at simple interest, and would be given by $£96{\cdot}115 \times {\cdot}03 \times {\cdot}1232 = £{\cdot}355$. So that the desired present value is $£96{\cdot}115 - £{\cdot}355 = £95{\cdot}760$, or £95 15s. 3d.

EXERCISE I

In answering these Questions the following logarithms and the tables at the end of the book may be used.

log 1·0035	·0015174	log 1·061679	·0259932
log 1·004034	·0017485	log 1·739278	·2403691
log 1·005	·0021661	log 2·090192	·3201862
log 1·012272	·0052973	log 3·29425	·5177566
log 1·0385	·0164065	log 5·646452	·7517757
log 1·0435	·0184925	log 8·234833	·9156549
log 1·0495	·0209824	log 9·81875	·9920562
log 1·05	·0211893		

1. Find the compound interest at an effective rate of $4\frac{1}{2}\%$ per annum on £276 5s. 6d. for 35 years.

2. Find the amount at an effective rate of interest of 3·85% per annum of £98 3s. 9d. in 20 years.

3. Find the present value at an effective rate of interest of $3\frac{1}{2}\%$ per annum of a sum of £25 7s. 6d. due 5 years hence.

4. Find the present value at an effective rate of interest of 4·35% per annum of a sum of £329 8s. 6d. due 15 years hence.

5. Find the compound interest on £124 7s. 9d. in 8 years at a nominal rate of interest of 6% per annum, convertible quarterly.

6. Find the discount, at a nominal rate of interest of 4·2% per annum, convertible monthly, upon a sum of £823 9s. 8d. due 9 years hence.

7. Find the effective rates of interest corresponding to the following nominal rates—

(*a*) 4% per annum payable quarterly ;
(*b*) $5\frac{1}{2}\%$ per annum payable half-yearly ;
(*c*) 6% per annum payable monthly.

8. (*a*) Find what nominal rate of interest, convertible quarterly, is equivalent to an effective rate of 5% per annum.

(*b*) Find what nominal rate of interest, convertible monthly, is equivalent to an effective rate of 4·95% per annum.

9. Find, approximately, the compound interest on £176 3s. 2d. for a period of 10 years 138 days at a nominal rate of interest of 8% per annum, payable with quarterly rests.

10. Find, approximately, the present value of £248 7s. 10d. due 5 years 88 days hence at a nominal rate of interest of 5% per annum payable half-yearly.

11. Find the bankers' discount on 15th March on a sum of £150 17s. 8d. due on the following 28th November, the rate to be employed being $5\frac{1}{2}\%$ per annum.

12. A building society issues fully paid "accumulative" investment shares under which no periodic payment of interest is made, but interest and bonus are added to the issue price until the share matures at the end of 10 years. A rate of interest of $3\frac{1}{2}\%$ per annum is guaranteed on the shares, and, in addition, bonuses are allotted out of the profits of the society. A shareholder whose share is maturing finds that the bonuses allotted amount to £12, and that the investment has yielded compound interest at the effective rate of 5% per annum. What sum did he originally invest, and what is the total maturity value of his share ?

CHAPTER II

ANNUITIES-CERTAIN

1. The subject of compound interest rests wholly upon the basis of the amount and present value of the individual sum, as discussed in the previous chapter. In fact, there is no reason why most problems in compound interest could not be solved without introducing any further conceptions. In practice, however, series of payments falling due at regular intervals so frequently arise that special methods have been devised for their treatment in calculation, mainly with a view to the saving of labour and time. All such series of payments are known as annuities, and when they are unconditionally payable throughout a fixed term of years, they are called annuities-certain. A practical instance of an annuity-certain is furnished by the dividends receivable from a Stock Exchange or similar security which is redeemable on a fixed date.

In the vast majority of instances, the periodic annuity-payments are equal, and, as will be seen later, may therefore be most conveniently dealt with in calculation by algebraical methods.

Geometric Progressions

2. In this connection, the branch of algebra which is of the utmost importance is that which deals with geometric progressions and their properties.

When the individual items of a series of numbers or quantities increase or decrease in the same proportion, they are said to be in geometric progression. The following are examples of such series—

$$3,\ 12,\ 48,\ 192,\ 768,\ \ldots$$

$$54,\ 18,\ 6,\ 2,\ 2/3,\ \ldots$$

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

The factor by which each item, or term, as it is called, must be multiplied so as to produce the next succeeding term is known as the common ratio: the common ratio may always, therefore, be ascertained by dividing any term by that which immediately precedes it. In the first of the above examples, the common ratio is 4, in the second it is 1/3, and in the third it is r.

The last of the above examples may be cited as the general case of a geometric series, since, by substituting in it appropriate values for a and r, any geometric progression may be derived; this particular series is, therefore, the only one which need be considered from the algebraic standpoint. For instance, if a were replaced by 54, and r by 1/3, the second of the above series would result.

It will be noted that the first term of the general series is a, the second term is ar, the third term is ar^2, the fourth term is ar^3, and so on, the index of r being in each instance one less than the number of the term in which it occurs. The fifth term would therefore be ar^4, the fourteenth term would be ar^{13}, and the m^{th} term would be ar^{m-1}.

It so happens that the relationship which exists among the terms of a geometric series makes it possible for the sum total of any number of terms to be found, without actual addition, and, in point of fact, it is not even necessary to calculate all of the terms themselves as a preliminary to their summation.

For example, suppose that it is desired to find the sum of the first m terms of the general series. If S is used to denote the required sum

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{m-2} + ar^{m-1}$$

(the index of r in the last, or m^{th} term, being $m - 1$)

Multiplying every item by the common ratio r,

$$rS = ar + ar^2 + ar^3 + \ldots + ar^{m-2} + ar^{m-1} + ar^m$$

By subtraction, therefore, $rS - S = ar^m - a$ (the terms from ar to ar^{m-1} inclusive vanishing).

$$\text{Whence } S = \frac{a(r^m - 1)}{r - 1} \qquad \qquad (12)$$

Alternatively, the result of the subtraction could have been stated as $S - rS = a - ar^m$

$$\text{Whence } S = \frac{a(1 - r^m)}{1 - r} \qquad \qquad (13)$$

It is simple to show that this formula is but a transposition of (12).

In practice it is more convenient to use formula (12) where the common ratio is greater than unity, and formula (13) where it is less than unity.

The following are examples of the application of the respective formulas—

EXAMPLE 1. To find the sum of the first 10 terms of the geometric series 5, 15, 45, 135 . . . .

Here the first term (corresponding to a in the general series) is 5, the common ratio (corresponding to r in the general series) is 3, and the number of terms to be summed (m) is 10.

The common ratio being greater than unity, formula (12) had better be employed

$$\text{Hence} \quad S = \frac{5(3^{10} - 1)}{3 - 1}$$

$$5(59{,}049 - 1)$$

$$= 147{,}620$$

EXAMPLE 2. To find the sum of the first 6 terms of the geometric series

2, 2/5, 2/25, 2/125 . . .

Here the first term (corresponding to a in the general series) is 2, the common ratio (corresponding to r in the general series) is 1/5, and the number of terms to be summed (m) is 6.

The common ratio being less than unity, formula (13) had better be employed.

$$\text{Hence} \quad S = \frac{2\left\{1 - \left(\frac{1}{5}\right)^6\right\}}{1} \text{ which simplifies to } 2\tfrac{1562}{3125}$$

These examples illustrate how it is possible to find the sum of any number of quantities in geometric progression (a) without actual addition, and (b) without calculating the individual quantities to be summed.

DEFINITIONS

3. Before proceeding to the problems presented by the valuation of annuities-certain, the following definitions must be given.

An annuity-certain is a series of periodical payments to be made unconditionally during a given length of time. In practice, the period between the successive annuity-payments would be a year, a half-year, a quarter, or some other part

of a year, but in theory it may be any interval whatever. The payments themselves are usually, but not necessarily, equal, and their aggregate in each year is called the "annual rent."

Annuities-certain may be divided into three classes, characterized by when the first annuity-payment falls due, viz.—

(*a*) *Immediate annuities*, under which the first annuity-payment falls due at the end of the first period.

(*b*) *Annuities-due*, under which the first annuity-payment falls due at the beginning of the first period.

(*c*) *Deferred annuities*, under which the first annuity-payment falls due more than one period hence.

A perpetuity is an annuity of which the payments are to continue for ever, as in the case of the dividends upon an irredeemable security. Strictly speaking, perpetuities are not annuities-certain at all, but it is convenient to study them concurrently with annuities-certain, like which they may be "immediate," "due," or "deferred" in type.

The present value of an annuity-certain is equal to the total of the present values of its respective payments; similarly, the amount of an annuity-certain is equal to the total of the accumulated amounts of the respective annuity-payments calculated to the end of the term for which the annuity is to run. Similarly, the present value of a perpetuity is equal to the total of the present values of the respective payments; it is true that the present value of each of the infinite number of payments cannot be separately ascertained by arithmetical methods, but their *total* may be found by algebraic methods. Amounts of perpetuities cannot, of course, come within the scope of practical work.

An annuity of which the present value is x times the annual rent is said to be "worth x years' purchase."

Amounts and Present Values

4. To find the *present value* at an effective rate of interest i of an *immediate* annuity-certain of 1 per annum payable yearly for n years.

The present value of such an annuity-certain is denoted symbolically by $a_{\overline{n}|}$. The present value itself is found by ascertaining the sum of the present values of the respective annuity-payments.

The present value of the first payment (due at the end of 1 year) is v.

The present value of the second payment (due at the end of 2 years) is v^2.

The present value of the third payment (due at the end of 3 years) is v^3.

And so on, the present value of the last payment (due at the end of n years) being v^n.

Hence $a_{\overline{n}|} = v + v^2 + v^3 + \ldots + v^n$

It will be seen that this series is a geometric progression whose first term is v, and whose common ratio is also v.

Since the common ratio (v) is less than unity, the sum of the n terms of the series will be found by applying formula (13).

So that $a_{\overline{n}|} = \dfrac{v(1 - v^n)}{1 - v}$

which, on multiplying numerator and denominator by $(1 + i)$, simplifies to $\dfrac{1 - v^n}{i}$.

$$\text{Thus} \qquad a_{\overline{n}|} = \frac{1 - v^n}{i} \qquad \ldots \qquad (14)$$

As an illustration of the application of this formula, suppose that it is desired to find the present value at an effective rate of interest of 5% per annum of an immediate annuity-certain of £28 7s. 6d. per annum payable yearly for 10 years.

The present value of each unit of annual rent being $a_{\overline{10}|}$, the present value of the annual rent of £28 7s. 6d. will be—

$$\begin{aligned} &£28{\cdot}375 \times a_{\overline{10}|} \\ &= £28{\cdot}375 \times \frac{1 - v^{10}}{\cdot} \\ \\ &= £28{\cdot}375 \\ \\ &= £28{\cdot}375 \times 7{\cdot}7217 \\ &= £219{\cdot}103 \\ &= £219 \text{ 2s. 1d. approx.} \end{aligned}$$

In practice, however, the calculation of the value of $\dfrac{1 - v^n}{i}$ is rarely necessary, since the values of that expression (or function) are tabulated in compound interest tables for all the

more usual rates of interest which arise, and for values of n from 1 up to 50 or 100 as the case might be. Should the rate of interest not be given in the tables, the expression would have to be evaluated by first finding the value of v^n by means of logarithms.

5. It may be convincing to show arithmetically that the value of $a_{\overline{10}|}$ as calculated above actually does represent the sum of the present values of the ten separate annuity-payments.

The present value of the	1st payment is	v	·95238
	2nd	v^2	·90703
	3rd		·86384
	4th	v^4	·82270
	5th		·78353
	6th		·74622
	7th		·71068
	8th		·67684
	9th		·64461
	10th		·61391
		TOTAL	7·72174

The saving in arithmetical work which is achieved by the use of the formula is clearly apparent, and the saving would have been still greater if the term of the annuity had exceeded 10 years.

6. To find the *amount* at an effective rate of interest i of an *immediate* annuity-certain of 1 per annum payable yearly for n years.

The amount of such an annuity-certain is denoted symbolically by $s_{\overline{n}|}$. The amount itself is found by ascertaining the sum of the amounts of the respective annuity-payments calculated to the end of the term of the annuity, viz. n years.

The first annual annuity-payment being due at the end of the first year will accumulate for the remaining $(n-1)$ years of the original term, the second annual annuity-payment will similarly accumulate for the remaining $(n-2)$ years, and so on; the last (i.e. the n^{th}) payment falling due at the end of the term will amount to 1 only, since there will remain no balance of the original term then unexpired.

Hence

but it will simplify the algebraic work if this series is regarded as being written in the reverse order, starting with 1 and

ending with $(1 + i)^{n-1}$. It will be seen that the series as rearranged is a geometric progression whose first term is 1 and whose common ratio is $(1 + i)$, which is clearly greater than unity.

$$+ i)^n - 1$$

As an illustration of the application of this formula, suppose that it is desired to find the amount, at an effective rate of interest of 5% per annum, of the immediate annuity-certain of £28 7s. 6d. per annum payable yearly for 10 years referred to in paragraph 4.

The amount of such an annuity-certain, if the annual rent were 1, would be $s_{\overline{10|}}$, but as the annual rent is £28 7s. 6d., the desired amount will be

$$£28{\cdot}375 \times s_{\overline{10|}} = £28{\cdot}375 \times \frac{(1+i)^{10}-1}{i}$$

$$= £28{\cdot}375 \times \quad {\cdot}05$$

$$= £28{\cdot}375 \times 12{\cdot}5779$$

$$= £356{\cdot}898$$

$$= £356\ 18s.\ 0d.\ \text{approx.}$$

In practice, however, the calculation of the value of $\frac{(1+i)^n - 1}{i}$ is rarely necessary, since the values of that function are tabulated in compound interest tables. Should the rate of interest not be given in the tables, the expression would have to be evaluated by finding the value of $(1 + i)^n$ by means of logarithms.

7. An arithmetical illustration of the identity of the results derived from the application of the formula for $s_{\overline{n|}}$ and from the actual addition of the amounts of the respective annuity-payments will now be given on the lines of paragraph 5.

Taking for this purpose $s_{\overline{10|}}$, calculated at an effective rate

of interest of 5% per annum, as in the case of the above example,

The amount of the	1st payment is	$(1+i)^9$	or	1·55133
	2nd ,,	$(1+i)^8$	,,	1·47746
	3rd ,,	$(1+i)^7$	,,	1·40710
	4th ,,	$(1+i)^6$	,,	1·34010
	5th ,,	$(1+i)^5$	,,	1·27628
	6th ,,	$(1+i)^4$	,,	1·21551
	7th ,,	$(1+i)^3$	,,	1·15763
	8th ,,	$(1+i)^2$	,,	1·10250
	9th ,,	$(1+i)$	,,	1·05000
	10th ,,	1	,,	1·00000
		TOTAL		12·57791

Here again the saving of work resulting from the use of the formula is evident.

8. An intimate connection always exists between the present value and the amount of an annuity-certain. Just as the present value represents the equivalent of the annuity-certain at the beginning of the term, so does the amount represent the equivalent of the annuity at the end of the term. The present value will, therefore, always accumulate to the amount by the end of the term, provided that the same rate of interest is used throughout the calculations. This result may be written symbolically

$$\times (1+i)^n = \qquad (16)$$

Similarly,

$$s_{\overline{n}|} \times v^n = a_{\overline{n}|} \qquad (17)$$

The correctness of these two formulas may be easily demonstrated algebraically by making the necessary substitutions for $a_{\overline{n}|}$ and $s_{\overline{n}|}$ in each case.

An arithmetical illustration based upon the values of $a_{\overline{10}|}$ and $s_{\overline{10}|}$ at an effective rate of 5%, as found above, will lend a reality to the relationship just discussed.

According to formula (16) the present value of the annuity (viz. 7·7217) should accumulate in 10 years to the amount of the annuity as found above, viz. 12·5779. Now 7·7217 will accumulate in 10 years to 7·7217 $(1·05)^{10}$ = 7·7217 × 1·62889. On multiplication this will be found to be 12·5779 as was to be expected.

Similarly $\quad a_{\overline{10}|} = s_{\overline{10}|} \times v^{10}$

$= 12{\cdot}5779 \times {\cdot}61391$

$= 7{\cdot}7217$ as before.

9. To find the *present value* at an effective rate of interest i of an *annuity-due* of 1 per annum payable yearly for n years.

In this case the first annuity-payment falls due immediately, and subsequent payments are made at yearly intervals thereafter, the last being due at the beginning of the nth year, or, what is the same thing, at the end of the $(n-1)$th year. This annuity-due is, therefore, equivalent to a sum of 1 (the first payment) together with an immediate annuity-certain of 1 per annum payable yearly for the next $(n-1)$ years.

The present value of an annuity-due of 1 per annum payable yearly for n years is denoted symbolically by $\mathrm{a}_{\overline{n}|}$, and consequently

$$\mathrm{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|} \quad . \quad . \quad . \quad (18)$$

The difference between the significances of a and a should be carefully noted.

Alternatively, the value of $\mathrm{a}_{\overline{n}|}$ could have been found by ascertaining the present values of the respective annuity payments and summing the resulting geometric series as was done in paragraph 4.

Thus $\mathrm{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \ldots + v^{n-2} + v^{n-1}$

$$= (1 + i) \times \frac{1 - v}{(1 + i) - 1}$$

on multiplying numerator and denominator by $(1 + i)$.

Thus an alternative formula is

$$\mathrm{a}_{\overline{n}|} = (1 + i) \times a_{\overline{n}|} \quad . \quad . \quad .$$

The equivalence of the two formulas for $\mathrm{a}_{\overline{n}|}$ may be simply shown as follows—

$$\mathrm{a}_{\overline{n}|} = 1 + \frac{1 - v^{n-1}}{i}$$

$$= (1 + i) \times a_{\overline{n}|}$$

It will now be shown how formula (19) could have been obtained by general reasoning. Consider an immediate annuity-certain of 1 per annum payable yearly for n years. At the beginning of the term it is equivalent in value to $a_{\overline{n}|}$, and at the end of the first year, just before the first annuity-payment is made, it will be equivalent in value to $a_{\overline{n}|}$ accumulated at interest for one year, i.e. to $(1 + i) \times a_{\overline{n}|}$. At the time when the first annuity-payment is due, however, the annuity itself constitutes an annuity-due, and the present value of this annuity-due must, therefore, be equivalent to $(1 + i) \times a_{\overline{n}|}$.

As will be seen later, this type of reasoning is extremely valuable, in that it enables many of the formulas entering into the subject of compound interest to be interpreted on a "common sense" basis.

10. To find the *amount* at an effective rate of interest of i of an *annuity-due* of 1 per annum payable yearly for n years.

Here, the first annuity-payment falls due immediately, and will therefore accumulate for the full term of n years. The second annuity-payment will accumulate for $(n - 1)$ years, the third for $(n - 2)$ years, and so on, the last payment accumulating for one year.

The amount of the annuity-due will therefore be equal to $(1 + i)^n + (1 + i)^{n-1} + (1 + i)^{n-2} + \ldots + (1 + i)^2 + (1 + i)$ Had there been another term, unity, at the end, this series would have represented the amount of an *immediate* annuity-certain of 1 per annum payable yearly for $n + 1$ years; the amount of the given annuity-due is therefore

$$s_{\overline{n+1}|} - 1 \quad . \quad . \quad . \quad . \quad (20)$$

Alternatively, the above series may be regarded as the result of multiplying the series given in paragraph 6 by $(1 + i)$, and since this latter series is equal to $s_{\overline{n}|}$ the present series is equal to $(1 + i) \times s_{\overline{n}|}$. This result corresponds to formula (19) for the present value of the annuity, and could evidently have been obtained by general reasoning, bearing in mind that each annuity-payment accumulates for one year more than would be the case with an immediate annuity.

It may be pointed out that there is no universally recognized symbol by which the *amount* of an annuity-due may be represented.

11. As will be appreciated from a consideration of formulas (18) and (20), the present values or amounts of annuities-due

at effective rates of interest may be found from compound interest tables by application of the following rules—

(*a*) To find the present value of an annuity-due of 1 per annum payable annually, take the present value of an *immediate* annuity of 1 per annum payable annually for a term one year shorter, and add unity thereto. For example, the present value, at a rate of interest of 5% per annum, of an annuity-due of 1 per annum payable annually for 15 years $= 1 + a_{\overline{14}|} = 1 + 9{\cdot}8986 = 10{\cdot}8986$.

(*b*) To find the amount of an annuity-due of 1 per annum payable annually, take the amount of an *immediate* annuity of 1 per annum payable annually for a term one year longer and deduct unity therefrom. For example, the amount, at a rate of interest of 4% per annum, of an annuity-due of 1 per annum payable annually for 26 years $= s_{\overline{27}|} - 1 = 47{\cdot}0842 - 1 = 46{\cdot}0842$.

The relationship existing between the amounts and present values of immediate annuities discussed in paragraph 8 evidently applies also to annuities-due.

12. To find the *present value* at an effective rate of interest i of a *deferred annuity-certain* of 1 per annum payable yearly for n years, deferred m years.

The present value of such a deferred annuity is symbolically denoted by $_{m|}a_{\overline{n}|}$. The first annuity-payment would fall due at the end of $(m + 1)$ years, and the last would fall due at the end of $(m + n)$ years.

The present value itself could, of course, be ascertained by finding the total of the present values of the individual annuity-payments, as was done in previous instances. It is, however, more convenient—and incidentally, more instructive—to rely upon methods of general reasoning. Two of such methods are available, viz.—

(*a*) If the annuity-payments were to be continued throughout the entire period of $(m + n)$ years, their present value would be $a_{\overline{m+n}|}$. In the present instance, however, the first m of such payments will not be received. Since the present value of the first m payments is $a_{\overline{m}|}$, the present value of the remaining n payments will therefore be $a_{\overline{m+n}|} - a_{\overline{m}|}$.

(*b*) At the end of the first m years the annuity-payments will constitute an immediate annuity of 1 per annum, payable yearly for n years, and their "present value" will then

be $a_{\overline{n}|}$. The value of these payments at the present moment must therefore be $v^m \times a_{\overline{n}|}$.

Hence $_{m|}a_{\overline{n}|} = v^m \times a_{\overline{n}|}$ . . . . . . (22)

It may easily be shown that these two forms are identical. Formula (21) would always be adopted in practice if compound interest tables were available, since it involves merely the subtraction of two tabulated values.

13. The *amount* of a *deferred annuity-certain* is the same as that of an immediate annuity-certain for the same number of payments; it is obvious that the period which has to elapse before the first payment is received under the deferred annuity does not enter into the calculations—it is the respective periods for which the payments are to be accumulated which are significant, and these are quite independent of the periods elapsing *before* the payments themselves fall due.

For this reason there is no symbol denoting the amount of a deferred annuity-certain.

14. To find the *present value* at an effective rate of interest i of an *immediate perpetuity* of 1 per annum payable yearly.

The present value of such a perpetuity is denoted symbolically by a_{∞}. Here "∞" replaces the "$\overline{n}|$" used in the case of an immediate annuity-certain, ∞ being the symbol employed to denote "infinity" in all mathematical work.

The formula for the present value of an immediate perpetuity may, perhaps, best be derived by considering what happens in the formula $a_{\overline{n}|} = \frac{1-v^n}{i}$ if the value of n be continuously increased. In such circumstances the value of the v^n in the numerator will successively diminish, since the present value of a unit of capital receivable n years hence becomes smaller as the numerical value of n is increased. If, therefore, the value of n be so increased that it is greater than any assignable quantity (i.e. until it is infinity), the value of v^n will become smaller than any assignable quantity and may be considered to have vanished altogether.

Thus, when n is made infinitely large, $a_{\overline{n}|}$ becomes $\frac{1}{i}$

Or, symbolically,

$$a_{\infty} = \frac{1}{i} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Alternatively, the present value of the perpetuity could be found by the following general reasoning.

Suppose that a unit of capital were invested at an effective rate of interest of i, and that it were allowed to remain invested indefinitely. The investor would receive an interest payment of i at the end of each year for ever, and there would be no question of the return of the original capital of 1 at the expiration of any fixed period. The interest payments of i thus receivable in perpetuity would be the equivalent of the unit of capital invested. In other words, the present value of a perpetuity of i per annum payable yearly would be 1. By simple proportion, therefore, the present value of a perpetuity of 1 per annum payable yearly would be $\frac{1}{i}$, as already found.

15. In conclusion of this chapter, similar methods of reasoning will be employed to find the values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ at an effective rate of interest.

(*a*) Suppose that a unit of capital were invested at an effective rate of interest of i, and that it were allowed to remain so invested for a total period of n years, being withdrawn at the end of that time. The investor would receive in exchange for a payment of 1—

(i) An immediate annuity of i per annum payable yearly for n years, representing the interest payments; and

(ii) the return of his unit of capital at the end of such n years.

The present value of the payments under (i) would be $i \times a_{\overline{n}|}$ and the present value of that under (ii) would be v^n, and the total of these would represent the original unit invested, i.e.—

$$i \times a_{\overline{n}|} + v^n = 1$$

Whence
$$a_{\overline{n}|} = 1 - v^n$$

(*b*) Suppose, as before, that a unit of capital were invested at an effective rate of interest of i for a period of n years. If the interest payments of i were themselves invested at compound interest at the effective rate of i per unit per annum, their accumulated amount by the end of the term of n years would be $i \times s_{\overline{n}|}$. But the total compound interest which would be earned from the investment of a unit of capital at an effective rate of interest of i per unit per annum for n years

Hence $$i \times s_{\overline{n}|} = (1 + i)^n - 1$$

And therefore $$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

EXERCISE II

In answering these Questions the following logarithms and the tables at the end of the book may be used.

log 1·0595	=	·0251010
log 1·112147	=	·0461620
log 8·99162	=	·9538380

1. Find the present value and the amount, at an effective rate of interest of $3\frac{1}{2}$% per annum, of an immediate annuity-certain of £26 7s. 7d. per annum, payable yearly for 28 years.

2. Find the present value and the amount, at an effective rate of interest of 5·95% per annum, of an immediate annuity-certain of £54 13s. 6d. per annum, payable yearly for 38 years.

3. Find the present value and the amount, at an effective rate of interest of $4\frac{1}{2}$% per annum, of an annuity-due of £64 16s. 7d. per annum, payable yearly for 16 years.

4. Find the present value and the amount, at an effective rate of interest of 5% per annum, of a deferred annuity-certain of £47 15s. 8d. per annum, payable yearly for 23 years, deferred 12 years.

5. An investor purchases an irredeemable security of nominal value £500 which yields dividends (in perpetuity) at the rate of 5% per annum payable yearly. What is the value of his income from the investment on the basis of an effective rate of interest of 6% per annum, assuming that the next dividend is due one year hence ?

6. Find the present value, at an effective rate of interest of 4% per annum, of an immediate annuity-certain payable yearly for 25 years, under which the annual rent is £125 for the first 7 years, and £150 for the balance of the term.

7. Given that at a rate of interest of 5% per annum $(1 + i)^n =$ 3·071524, find the value of $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

8. A lease has been granted for a term of 12 years at an annual rent of £150. Five years of the lease have expired, and the holder desires to extend its term for five years beyond the date originally fixed for its expiry. If the value of the property is such that an annual rental of £200 per annum could be obtained after the expiry of the original lease, find, to the nearest £, what sum should be paid down at the present time for the right of extending the lease for the further five years at the present rental of £150 per annum. The calculation should be made on the basis of an effective rate of interest of 5% per annum.

9. Find what annual sum which, if invested at compound interest at the effective rate of $4\frac{1}{2}$% per annum, would amount to £325 in 25 years, it being assumed that such annual sums are invested at the end of each year.

10. The present value of an immediate annuity-certain of 1 per annum payable yearly for a certain number of years is 29·0800, and the present value, at the same effective rate of interest, of an annuity-due of 1 per annum payable yearly for the same number of years is 29·6616. Upon what effective rate of interest are these present values based ?

CHAPTER III

ANNUITIES-CERTAIN (*continued*)

1. In the previous chapter it was shown how to find the present values and amounts of annuities-certain which are payable annually. In practice, however, annuities are frequently payable more often than annually, and it is, therefore, important to investigate the relative formulas in these cases.

2. To find the *present value* at an effective rate of interest i of an *immediate* annuity-certain of 1 per annum payable p times a year (in equal instalments of $1/p$) for n years.

The present value of such an annuity-certain is denoted symbolically by $a^{(p)}_{\overline{n}|}$. The present value itself is found by the usual method of ascertaining the sum of the present values of the respective annuity payments.

The present value of the first payment of $1/p$ due at the end of $1/p^{\text{th}}$ of a year is $\frac{1}{p}\, v^{\frac{1}{p}}$.

The present value of the second payment of $1/p$ due at the end of $2/p^{\text{ths}}$ of a year is $\frac{1}{p}\, v^{\frac{2}{p}}$.

And so on, the present value of the last payment, due at the end of n years, being $\frac{1}{p}\, v^{n}$.

$$\text{Hence } a^{(p)}_{\overline{n}|} = \frac{1}{p}\left(v^{\frac{1}{p}} + v^{\frac{2}{p}} + v^{\frac{3}{p}} + \ldots + v^{n}\right)$$

$$\begin{array}{c} 1 - v^{\frac{}{p}} \\ \dfrac{1 - v^{n}}{1} \end{array} \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

It will be seen that in this formula the denominator is equal to the nominal rate of interest convertible p times a year corresponding to the effective rate of interest (i) at which the

valuation is being made (see formula (11)) and that the numerator is the same as when the annuity is payable annually (see formula (14)).

Formula (24) may therefore be transposed to

$$a^{(p)}_{\overline{n}|} = \frac{i}{j_{(p)}} \times a_{\overline{n}|} \quad . \quad . \quad . \quad (25)$$

but it must be carefully borne in mind that $j_{(p)}$ is here used as a contraction for $p\{(1+i)^{\frac{1}{p}} - 1\}$ and does not actually introduce a nominal rate of interest into the valuation itself.

If, then, the values of $\frac{i}{j_{(p)}}$ are tabulated for the values of p likely to arise in practice, viz. 2, 4, and 12, corresponding to half-yearly, quarterly, and monthly payments of annuity, a single multiplication will enable $a^{(p)}_{\overline{n}|}$ to be obtained from the tabulated value of $a_{\overline{n}|}$.

It is to be observed that this multiplying factor depends only upon the effective rate of interest employed in the valuation and the frequency of the annuity payments; it is quite independent of the term for which the annuity is to run.

As an illustration of the application of formula (25), consider the example given in paragraph 4 of Chapter II, and assume that the annuity were payable in equal half-yearly portions of £14 3s. 9d.

The present value of each unit of annual rent being $a^{(2)}_{\overline{10}|}$ the present value of the annual rent of £28 7s. 6d. will be—

$$£28{\cdot}375 \times a^{(2)}_{\overline{10}|}, \text{ calculated at } 5\%$$

$$= £28{\cdot}375 \times \frac{i}{j_{(2)}} \times a_{\overline{10}|},$$

which, on taking the necessary values from the tables,

$$= £28{\cdot}375 \times 1{\cdot}01235 \times 7{\cdot}7217$$
$$= £221{\cdot}809$$
$$= £221 \text{ 16s. 2d., approx.}$$

It will be seen that the present value of the annuity is greater when the payments are made half-yearly than when they are made yearly; this must obviously be the case, since one half of the annual rent is received six months earlier in each year when the payments are made half-yearly.

3. To find the *amount* at an effective rate of interest i of an *immediate* annuity-certain of 1 per annum, payable p times a year (in equal instalments of $1/p$) for n years.

The amount of such an annuity-certain is denoted symbolically by $s^{(p)}_{\overline{n|}}$. The amount itself is found by the usual method of ascertaining the sum of the amounts of the respective annuity payments.

Working on these lines, which have already been illustrated, it will be found that

$$s_{\overline{n|}} = \frac{\quad}{p\{(1+i)^{\frac{1}{p}} - 1\}} \quad . \quad . \quad .$$

This formula may be transposed similarly to formula (24), and, in passing, it may be noted that $s^{(p)}_{\overline{n|}} = (1+i)^n \times a^{(p)}_{\overline{n|}}$ and $a^{(p)}_{\overline{n|}} = v^n \times s^{(p)}_{\overline{n|}}$, which relationships correspond with those of formulas (16) and (17).

The transposition just referred to would give

So that, to find the amount at an effective rate of interest i of an immediate annuity-certain payable p times a year, it is merely necessary to multiply the amount (at the same effective rate of interest) of the corresponding *yearly* annuity by the value of $\frac{i}{j_{(p)}}$ derived from the compound interest tables.

It is important to note, however, that formulas (25) and (27) apply only in cases where the valuation is to be made at an *effective* rate of interest.

4. The *present value* at an effective rate of interest i of an *annuity-due* of 1 per annum payable p times a year (in equal instalments of $1/p$) for n years may be found by two methods, viz.—

(*a*) By adding the first payment to the present value of the remaining payments; thus

$$\text{a}^{(p)}_{\overline{n|}} = \frac{1}{p} + a^{(p)}_{\overline{n-\frac{1}{p}|}} {}^{*} \quad . \quad . \quad .$$

and (*b*) by accumulating the present value of the corresponding immediate-annuity for one payment-period; thus

$$\text{a}_{\overline{n}|}^{(p)} = (1+i)^{\frac{1}{p}} \times a_{\overline{n}|}^{(p)} \quad . \quad . \quad (29)$$

As was seen in paragraph 11 of Chapter II, it is more convenient to use the former method when the annuity is payable annually, i.e. when $p = 1$; when, however, the annuity is payable more frequently than annually, it is more convenient to use method (*b*), as it avoids the calculation of an annuity value for a fractional term of years. In the application of method (*b*) the value of $(1+i)^{\frac{1}{p}}$ necessary to effect the accumulation for $\frac{1}{p}$th of a year would be calculated by means of logarithms, if it were not given in the tables.

5. The *amount* at an effective rate of interest i of an *annuity-due* of 1 per annum, payable p times a year (in equal instalments of $1/p$) for n years, may also be found by two methods, viz.—

(*a*) By taking the amount of an immediate annuity for a term one payment-period longer, and deducting therefrom the final annuity payment of $1/p$; thus

$$(p) \quad * \quad - \frac{1}{p} \quad . \quad . \quad . \quad (30)$$

and (*b*) by accumulating for one payment-period the amount of the corresponding immediate annuity; thus

$$(1+i)^{\frac{1}{p}} \times s_{\overline{n}|}^{(p)} \quad . \quad . \quad . \quad (31)$$

As stated in the previous chapter, there are no symbols appropriate to amounts of annuities-due corresponding to those of the type $\text{a}_{\overline{n}|}$ employed for present values.

As in the case of present values, it is more convenient to use the former method when the annuity is payable annually, i.e. when $p = 1$; when, however, the annuity is payable more frequently than annually, it is more convenient to use the second method.

6. The formulas for the *present value* of a *deferred* annuity-certain, payable more frequently than annually, may be

$$* \; . \; ^{(p)} \; . \; \_ \quad i \quad \times \; (1+i)^{n+\frac{1}{p}} - 1$$

derived by methods parallel to those discussed in paragraph 12 of Chapter II, the formulas themselves being

$$ {}_{m|}a^{(p)}_{\overline{n}|} = a^{(p)}_{\overline{m+n}|} - a^{(p)}_{\overline{m}|} \qquad . \quad . \quad (32)$$

and
$$ {}_{m|}a^{(p)}_{\overline{n}|} = v^{m} \times a^{(p)}_{\overline{n}|} \qquad . \quad . \quad (33)$$

7. From formula (24) and a consideration of paragraph 14 of the previous chapter it is clear that the *present value* at an effective rate of interest i of an *immediate perpetuity* of 1 per annum payable p times a year (in equal instalments of $1/p$) is

$$\frac{1}{\qquad} \text{ or } \frac{1}{\quad}$$

Whence $a^{(p)}_{\infty} = \dfrac{1}{j_{(p)}}$ . . . . (34)

It is interesting to observe that the present value could also be found by applying the principle underlying formula (25) thus—

$$a^{(p)}_{\infty} = \frac{i}{\quad} \times a_{\infty}$$

$$j_{(p)} \qquad i$$

$$= \frac{1}{j_{(p)}} \text{ as before.}$$

Nominal Rate of Interest

8. Up to the present, the rate of interest employed in calculating the present values and amounts of the various types of annuity-certain has been taken as an effective rate. Sometimes, however, calculations have to be made at a nominal rate of interest, convertible with a specified frequency. To deduce formulas applicable to a nominal rate of interest, all that would be necessary would be to proceed on the usual lines of summing the present values or amounts of the respective annuity payments, expressing each of such present values or amounts in terms of the nominal rate of interest involved (see formulas (5) and (8)).

It will be found, however, that the only effect upon the formulas as applicable to an effective rate of interest is to substitute for i the total amount of compound interest which would be earned in a year at the given nominal rate. For

instance, if the valuation were to be made at a nominal rate of interest of j convertible m times a year, i would be replaced throughout by $\left(1+\frac{j}{m}\right)^m - 1.$

The *amount*, at a nominal rate of interest of j convertible m times a year, of an *immediate* annuity-certain of 1 per annum payable p times a year for n years would therefore be given by

since i would be replaced in each instance by $\left(1+\frac{j}{m}\right)^m - 1,$ and, consequently, $\left(1+\frac{j}{m}\right)^m$ would be substituted throughout for $(1+i)$. In the case of present values the same procedure would apply, v being replaced by $\left(1+\frac{j}{m}\right)^{-m}$, and v^n being correspondingly changed to $\left(1+\frac{j}{m}\right)^{-mn}$

It might appear at first sight that the evaluation of results from the formulas thus obtained would be troublesome, but in most instances the use of compound interest tables will greatly assist in the calculations, as the following example will show.

To find the *present value* of an *immediate* annuity-certain of 1 per annum payable yearly for 10 years, using a nominal rate of interest of 5% per annum convertible half-yearly.

At an effective rate of interest of i, the present value would be $\frac{1-(1+i)^{-10}}{i}$, and hence at a nominal rate of interest of j per unit per annum payable half-yearly, the present value would be

$$\frac{1-\left(1+\frac{j}{2}\right)^{-}}{\left(1+\frac{j}{2}\right)^{2}-}$$

In this instance the value of j is ·05, so that the desired present value would be given by

$$\frac{1-(1\cdot025)^{-20}}{(1\cdot025)^{2}-1}$$

Each of the powers of (1·025) could be evaluated by reference to compound interest tables, that in the numerator representing the present value of a unit of capital due 20 years hence, and that in the denominator representing the amount of a unit of capital accumulated for a period of 2 years; the appropriate tables would, of course, be those corresponding to a rate of interest of $2\frac{1}{2}\%$ per annum.

The value of the above expression would thus be

$$\frac{1-\cdot61027}{1\cdot05063-1}$$

$$=\frac{\cdot38973}{\cdot05063}$$

$$=7\cdot698$$

9. It frequently happens in practice that the rate of interest to be used is convertible with the same frequency as that with which the annuity is payable. Such a case would be presented by the valuation of $a^{(p)}_{\overline{n}|}$ at a nominal rate of interest of $j_{(p)}$. Making the valuation by means of formula (24), suitably modified owing to the substitutions necessitated by the use of a nominal rate of interest, it will be found that

$$a^{(p)}_{\overline{n}|} \qquad 1-\left(1+\frac{j}{p}\right)^{-pn}$$

$$p \times \qquad \frac{j}{p}$$

calculated at an effective rate of interest of $\frac{j}{p}$.

Similarly, it may be shown that $s_{\overline{n}|}^{(p)}$ calculated at a nominal rate of interest of $j_{(p)} = \frac{1}{p} \times s_{\overline{np}|}$, calculated at an effective rate of interest of $\frac{j}{p}$.

Quite apart, however, from any algebraic demonstration, it is evident that, at a nominal rate of interest of $j_{(p)}$, an annuity-certain of $\frac{1}{p}$ per interval payable for np intervals has a present value of $\frac{1}{p} \times a_{\overline{np}|}$ and an amount of $\frac{1}{p} \times s_{\overline{np}|}$, the calculations being made at an "effective" rate of interest of $\frac{j}{p}$ per interval. For example, if the annuity were one of £40 per annum payable half-yearly for 5 years, the nominal rate of interest being 5% per annum payable half-yearly, the present value would be £20 $\times\, a_{\overline{10}|}$, based on an effective rate of interest of $2\frac{1}{2}$%, which, from compound interest tables, will be found to be £175·042.

In this connection it is important to note that in the case of all annuities-certain the product of the suffix and the coefficient of a or s equals the total of the cash payments under the annuity. Thus, in the above formulas $np \times \frac{1}{p} = n$, and in the example 10 × £20 = 5 × £40. Reference is made to this point because it is a common fault for beginners to overlook the coefficient $\frac{1}{p}$, when working practical examples which involve a use of the artifice herein referred to.

In conformity with the principle discussed in the present paragraph, the present value at a nominal rate of interest of $j_{(p)}$ of an *annuity-due* of 1 per annum payable p times a year for n years would be given by

'1 . 1

calculated at an effective rate of interest of $\frac{j}{p}$; the corresponding amount would be given by $\frac{1}{p} \times s_{\overline{np+1}|} - \frac{1}{p}$.

10. The case where the annuity is payable with a different frequency from that with which the nominal rate of interest is

convertible arises very rarely in practice, but it sometimes happens that the frequency with which the annuity is payable is a *multiple* of that with which the interest is convertible.

In these latter instances, the calculations may be considerably shortened by adopting the device mentioned in the last paragraph in conjunction with that described at the end of paragraph 2 of this chapter. An example will show best how this can be done.

Suppose that it is required to find the present value of an immediate annuity of £50 per annum payable quarterly for 10 years at a nominal rate of interest of 5% per annum convertible half-yearly. The present value may be written down, in accordance with the principle explained in the last paragraph, as £25 $\times\ a^{(2)}_{\overline{20}|}$, calculated at an effective rate of interest of $2\frac{1}{2}\%$ per annum, because (*a*) the total of the annuity payments received in each interval of half-year is £25 ; (*b*) the number of annuity payments in each interval is 2 ; (*c*) the number of half-yearly intervals for which the annuity is payable is 20 ; and (*d*) the rate of interest operating in each interval is $2\frac{1}{2}\%$.

The value of $a^{(2)}_{\overline{20}|}$ at $2\frac{1}{2}\%$ is, however, given by $\dfrac{i}{j_{(2)}} \times a_{\overline{20}|}$ at $2\frac{1}{2}\%$, or $1{\cdot}00621 \times 15{\cdot}5892$, so that the desired present value of the original annuity is £25 $\times\ 1{\cdot}00621 \times 15{\cdot}5892$. It will be seen that the calculations have been performed without the necessity for deriving a formula by the method of substitution referred to in paragraph 8 of this chapter.

11. In conclusion of this chapter, it must be pointed out that the symbols $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, etc., give no indication either of the rate of interest involved, or of the frequency of its conversion.

EXERCISE III

In answering these Questions the tables at the end of the book may be used.

1. Find the present value and the amount, at an effective rate of interest of 4% per annum, of an immediate annuity-certain of £25 3s. 6d. per annum, payable half-yearly for 20 years.

2. Find the present value and the amount, at an effective rate of interest of 5% per annum, of an annuity-certain of £12 8s. 4d. per annum, payable half-yearly in advance for 10 years.

3. (*a*) Find the present value, at an effective rate of interest of $3\frac{1}{2}\%$ per annum, of an annuity-certain of £35 18s. 6d. per annum, payable quarterly for 10 years, under which the first payment falls due at the end of $5\frac{1}{4}$ years.

(*b*) How would the result be affected if the first payment fell due at the end of 5 years ?

4. (*a*) Find the value, at an effective rate of interest of 5% per annum, of a perpetual ground rent of £40 per annum, payable half-yearly, the first payment of which falls due 3 months hence.

(*b*) What would the value be, at a nominal rate of interest of 5% per annum, convertible half-yearly ?

5. Find the present value and the amount, at a nominal rate of interest of 4% per annum, convertible half-yearly, of an immediate annuity-certain of £45 10s. 0d. per annum, payable half-yearly for 20 years.

6. Find the present value and the amount, at a nominal rate of interest of 5% per annum convertible half-yearly, of an immediate annuity-certain of £48 per annum, payable quarterly for 10 years.

7. Find the present value and the amount, at a nominal rate of interest of 6% per annum convertible quarterly, of an immediate annuity-certain of £10 per annum, payable half-yearly for 8 years.

8. Given that, at an effective rate of interest of 5% per annum, the present value of an immediate annuity-certain, payable yearly, is £85 7s. 8d., find what the present value would be if the annuity were payable quarterly.

CHAPTER IV

VALUATION OF REDEEMABLE SECURITIES

1. A REDEEMABLE security is one under which interest is payable for a specified period of years, at the end of which time the nominal capital value is repaid, possibly with some small addition. The nominal capital value is the amount upon which interest (or dividends) is calculated, and this may be either more or less than the price at which the security was originally issued to the public. When the issue is made at a price which is less than the nominal value, the security is said to be issued at a discount; when it is issued at a price exceeding the nominal value, it is said to be issued at a premium. The amount which is repaid to the holder at maturity is called the redemption value; this value may differ from the issue price and the nominal value, although it would not usually be less than either of these. A security which is redeemable at a price in excess of its nominal value is said to be redeemable at a premium; conversely, if a security were redeemable at a price smaller than its nominal value it would be said to be redeemable at a discount, but cases of this kind do not arise in practice.

2. The valuation of redeemable securities constitutes one of the most important practical applications of the theory of compound interest. The problem of valuation is to find what amount a purchaser should give for the prescribed security, so as to realize a stated rate of compound interest from the investment. For example, take the case of a debenture of nominal value £100, which is redeemable in 5 years at £102, and which bears interest at the rate of 5% per annum, payable yearly. An investor might be willing to purchase this security, if he could realize an effective yield of 4% per annum from his investment; the problem is, what price should he give in order to ensure such a return?

The first point to observe is that a purchaser would receive—

(*a*) £102 at the end of 5 years; and

(*b*) An immediate annuity-certain of the annual interest of 5% on the nominal value for 5 years. (It is assumed that the first interest payment to be received would be due at the end of one year.)

If, therefore, he desires a yield of 4% per annum from the investment, he should pay a price equal to the sum of the present values of (*a*) and (*b*), calculated at that rate of interest.

The required purchase price is thus

$$£102 \times v^5 + £5 \times a_{\overline{5}|},$$

calculated at an effective rate of interest of 4% per annum,

$$= £102 \times \cdot 82193 + £5 \times 4\cdot 4518$$
$$= £106\cdot 096$$
$$= £106 \text{ 1s. 11d., approximately.}$$

3. It is instructive to show how the investor purchasing the security at this price will actually derive the desired yield. This may be done as follows—

	£
Capital originally invested . . .	106·096
Interest at 4% thereon . . .	4·244
	110·340
Deduct first dividend	5·000
Capital standing invested at end of 1st year	105·340
Interest at 4% thereon . . .	4·214
	109·554
Deduct second dividend . . .	5·000
Capital standing invested at end of 2nd year	104·554
Interest at 4% thereon . . .	4·182
	108·736
Deduct third dividend	5·000
Capital standing invested at end of 3rd year	103·736
Interest at 4% thereon . . .	4·149
	107·885
Deduct fourth dividend . . .	5·000
Capital standing invested at end of 4th year	102·885
Interest at 4% thereon . . .	4·115
	107·000
Deduct fifth dividend	5·000
Capital standing invested at end of 5th year	102·000

which is repaid by the redemption value of £102 at maturity.

4. The case of the valuation of a redeemable security will now be considered from the algebraic standpoint.

Let R = the redemption value of the security,

n = the number of years at the end of which it is redeemable,

and D = the annual interest (dividend) payable thereon p times a year in equal instalments of $\frac{D}{p}$, the first instalment being due at the end of $\frac{1}{p}$th of a year.

Then, the present value of the security will be

$$R \times v^n + D \times a^{(p)}_{\overline{n}|} \quad . \quad . \quad . \quad (35)$$

In applying this formula, the calculations would be made at the rate of interest which the investor wishes to secure from the investment.

It is, of course, quite possible that the investor would require to earn a *nominal* rate of interest, and, if so, the present values of the redemption value and the annuity-certain representing the dividends would be ascertained by the appropriate methods discussed in previous chapters.

5. An alternative method of valuation is available, which is frequently easier to apply in practice, particularly if the security is redeemable at its nominal value. It will be sufficient to illustrate this method on the assumption that the valuation is required at an effective rate of interest i.

As shown in the previous paragraph, the desired present value of the security will be

$$R \times v^n + D \times a^{(p)}_{\overline{n}|},$$

calculated at an effective rate of interest of i. Now, since

$$1 - v^n$$

it may easily be shown that $v^n = 1 - i\, a_{\overline{n}|}$. On substituting this value for v^n in the above formula, and changing $a^{(p)}_{\overline{n}|}$ into its equivalent $\frac{i}{i^{(p)}} \times a_{\overline{n}|}$ the result becomes

$$(36)$$

where g represents $\frac{D}{R}$, the ratio of the annual dividend to the redemption value.

Hence, to value any redeemable security at an effective rate of interest i, the redemption value must be multiplied by

$$1 + \left(g \cdot \frac{i}{j_{(p)}} - \right.$$

As an example of the application of this method, a debenture of nominal value £100 redeemable 10 years hence at £105, bearing interest at 6% per annum, payable quarterly, will be valued at an effective rate of interest of 5%.

$$\text{Here } R = 105, \; D = 6, \; g = 6/105 = \cdot057143,$$

$$\frac{i}{j_{(p)}} = 1\cdot01856, \text{ and } a_{\overline{n}|} = 7\cdot7217$$

Therefore, the value of the debenture on an effective 5% interest basis is

$$105 \times \{1 + (\cdot057143 \times 1\cdot01856 - \cdot05) \times 7\cdot7217\}$$
$$= £111\cdot651.$$

It will be noted that if the security were redeemable at par (i.e. at its nominal value) the value of g would be obvious without calculation, and in such a case the method requires the minimum of labour, only two values having to be extracted from the interest tables.

6. It sometimes happens that the numerical value of $(g \cdot \frac{i}{j_{(p)}} - i)$ is negative, but in such cases formula (36) may still be applied; the result of the negative value will necessarily be to produce a value of the security which is less than the redemption value. Thus, if the valuation is made at an effective rate of interest which is greater than $g \cdot \frac{i}{j_{(p)}}$ the present value of the security will stand at a discount on its redemption value. Conversely, if the valuation is made at an effective rate of interest which is less than $g \cdot \frac{i}{j_{(p)}}$, the present value of the security will stand at a premium on its redemption value.

If k be employed to denote the amount of such premium in respect of each unit of the redemption value, the following formula will apply—

$$k = (g \cdot \frac{i}{j_{(p)}} - i) \times a_{\overline{n}|}$$

When the numerical result of this formula is negative, a discount on the redemption value is represented.

7. If dividends are payable yearly, formula (36) becomes

which may be easily adapted to meet all cases where the valuation rate of interest is payable with the same frequency as the dividends themselves.

Suppose, for example, that interest under the debenture valued in paragraph 5 had been payable half-yearly instead of quarterly, and that its value had been required to yield 5% per annum, also convertible half-yearly. If the unit of time-measurement be changed from a year to a half-year, the unexpired term of the debenture would be 20 intervals, the dividend per interval would be 3%, and the valuation rate of interest would be $2\frac{1}{2}$% per interval, effective. The redemption value and the nominal value of the debenture would, of course, be unaffected by the change in the time-unit. The factors entering into the valuation would thus be—

$$R = 105,\ D = 3,\ g = 3/105 = \cdot 028571,\ n = 20,$$
$$i = \cdot 025 \text{ and } a_{\overline{20}|} \text{ at } 2\tfrac{1}{2}\% = 15\cdot 5892.$$

The value of the debenture would therefore be

$$105\ \{1 + (\cdot 028571 - \cdot 025) \times 15\cdot 5892\} = £110\cdot 845$$

It may be added that in practice it is quite common to value redeemable securities on the basis of a nominal rate of interest, convertible with the same frequency as that with which the dividends are payable.

Sinking Funds

8. Where a redeemable security has been purchased at a premium on its redemption value, there would be a capital loss at maturity if the purchaser regarded the whole of the dividends as interest on the purchase price.

This capital loss would be more apparent than real, because if the dividends were so regarded as interest, they would give a greater return on the purchase price than was anticipated in the original valuation. Should, however, the purchaser adopt the correct practice of taking credit for interest at the exact valuation rate, there would remain a balance of each periodic dividend, and these balances would, if accumulated at compound interest at the valuation rate, ultimately make good the difference between the purchase price and the redemption value. This point may be conveniently illustrated

by means of the example in paragraph 2. Here, the purchaser pays £106·096 for a debenture of nominal value £100, redeemable in 5 years at £102, and bearing yearly interest of 5% per annum, and he thereby obtains an effective yield of 4% per annum from the investment. The premium on the redemption value in this instance is £4·096. Interest at 4% on the purchase price is £4·244, and, the yearly dividend being £5, there remains a balance each year of £·756. These balances would accumulate at the valuation rate of interest of 4% per annum to $£·756 \times s_{\overline{5}|} = £·756 \times 5·4163 = £4·096$ by the maturity of the debenture, and would thus replace the premium on the redemption value. These balances of dividend are described as sinking fund payments, and are, as has been shown, such periodic amounts as will, with compound interest accumulations, exactly suffice to replace by maturity the excess of the purchase price over the redemption value.

In practice, however, these balances of dividend would customarily be aggregated with any other sums which the investor might have available for investment. It is not usual, therefore, actually to set up a separate sinking fund, but when each dividend is received, the capital value of the security is written down in the investment ledger, to an amount equal to the original purchase price, less the amount to which the sinking fund would then have accumulated.*

In the present instance, for example, the sinking fund accumulations would amount at the end of successive years to the following sums—

Year.	Total amounts of sinking fund accumulations at end of year. £		£
1	$·756\ s_{\overline{1}|}$	=	·756
2	$·756\ s_{\overline{2}|}$	=	1·542
3	$·756\ s_{\overline{3}|}$	=	2·360
4	$·756\ s_{\overline{4}|}$	=	3·211
5	$·756\ s_{\overline{5}|}$	=	4·096

After writing these respective amounts off the original

* This method of treatment, whilst theoretically correct, would not necessarily be adopted in practice, where other considerations would enter. For instance, in order to avoid the possibility of a serious capital loss arising if the security should be sold at any time, the capital value might be assessed in the ledger year by year at the current Stock Exchange middle price.

value of the security, the ledger values at the end of successive years would be—

Year.	Ledger value of security at the end of the year.
1	£105·340
2	104·554
3	103·736
4	102·885
5	102·000

It will be seen that these latter figures are identical with those shown in the statement in paragraph 3 as the capital standing invested at the end of successive years.

It may be added that these capital values are also identical with those which would result from the re-valuation of the security at the original rate of interest; this is demonstrated algebraically in the Appendix.

Negative Sinking Funds

9. If a redeemable security is purchased at a discount on its redemption value, a capital profit would be shown at maturity, if the investor regarded the dividends as the interest on the purchase price. It is to be observed, however, that the dividends alone would not give the investor the yield he anticipated when he purchased the security. Besides taking credit for the dividends as interest, he should gradually write up the ledger value of the security by the respective amounts by which the dividends fall short of the interest on the current ledger values.* In this way, what is technically known as a negative sinking fund arises. It is obviously impossible actually to set up a negative sinking fund, as it represents the accumulation of sums which have not been received; the only possible course is to write up the ledger value of the security from time to time in the manner indicated, so that the ledger value will correspond with a re-valuation of the security at the original valuation rate of interest.

For example, suppose that a debenture of nominal value £100 is redeemable at the end of 6 years at £103, and that it bears a rate of interest of 4% per annum, payable half-yearly. The purchase price to yield 5% per annum, convertible half-yearly, would be $103\ v^{12} + 2\,a_{\overline{12}|}$ calculated at an effective rate of interest of $2\frac{1}{2}$%, or £97·103.

Now, a half-yearly dividend of £2 in respect of the investment

* See footnote on page 40.

of £97·103 does not represent a return of 2½% per half-year. But, since the investment does actually yield 2½% per half-year, the amount by which 2½% of the purchase price exceeds the half-yearly dividend of £2 (i.e. £·4276) may be taken credit for as a half-yearly negative sinking fund payment, the accumulations of which (at the valuation rate of interest of 2½% per half-year) should be added to the ledger value of the security from half-year to half-year. These sinking fund accumulations would amount at the end of successive half-years to the following sums—

Half-year.	Total amounts of sinking fund accumulations at end of half-year. £	£
1	$\cdot 4276\, s_{\overline{1}\rceil}$ =	·428
2	$\cdot 4276\, s_{\overline{2}\rceil}$ =	·866
3	$\cdot 4276\, s_{\overline{3}\rceil}$ =	1·315
4	$\cdot 4276\, s_{\overline{4}\rceil}$ =	1·776
5	$\cdot 4276\, s_{\overline{5}\rceil}$ =	2·248
6	$\cdot 4276\, s_{\overline{6}\rceil}$ =	2·731
7	$\cdot 4276\, s_{\overline{7}\rceil}$ =	3·227
8	$\cdot 4276\, s_{\overline{8}\rceil}$ =	3·736
9	$\cdot 4276\, s_{\overline{9}\rceil}$ =	4·257
10	$\cdot 4276\, s_{\overline{10}\rceil}$ =	4·791
11	$\cdot 4276\, s_{\overline{11}\rceil}$ =	5·338
12	$\cdot 4276\, s_{\overline{12}\rceil}$ =	5·897

After writing up the original value of the security by these respective amounts, the ledger values at the end of successive half-years would be—

Half-year.	Ledger value of security at the end of the half-year.
1	£ 97·531
2	97·969
3	98·418
4	98·879
5	99·351
6	99·834
7	100·330
8	100·839
9	101·360
10	101·894
11	102·441
12	103·000

These latter amounts are those which would result from the re-valuation of the security at the original valuation rate, viz., 5% per annum, convertible half-yearly. As mentioned in the last paragraph, this proposition is demonstrated algebraically in the Appendix.

10. When a redeemable security is purchased at a *premium* on its redemption value, it might, in special circumstances, be necessary actually to set up a separate sinking fund. In such a case it has to be considered whether the sinking fund could, in fact, be invested at the rate of interest yielded by the investment itself; it might conceivably happen that it could be invested only at a lower rate.

An elementary algebraic example will suffice to illustrate the point.

Let i' = the effective rate of interest at which the sinking fund can be invested.

i = the effective rate of interest which is to be yielded by the investment.

x = the annual sinking fund instalment required to replace the premium on the redemption value.

And V = the desired value of the security.

The same meanings are attached to the remaining symbols here employed as in the earlier portions of this chapter. The premium on the redemption value will be $V - R$; and this premium must be equal to the sinking fund accumulations as at the end of the term of n years.

Hence $x \times s'_{\overline{n}|} = V - R$,

the accent to s denoting that a rate of interest of i' is to be employed.

Therefore $\qquad x = (V - R) \div s'_{\overline{n}|}$

and the balance of the yearly dividend after deduction of the sinking fund instalment

Since, then, an effective rate of interest of i is desired from the investment, this last quantity must equal iV.

Thus $$iV = D - \frac{V - R}{} \quad . \quad . \quad . \quad . \quad (37)$$

A solution of this equation gives

$$V = \frac{R + D \,.\, s'_{\overline{n|}}}{1 + i \,.\, s'_{\overline{n|}}} \quad . \quad . \quad . \quad . \quad (38)$$

As an illustration of the application of this formula, take the example in paragraph 2 of this chapter, assuming that the purchaser required to earn 4% per annum, as before, but could reinvest the sinking fund instalments at only 3% per annum.

In this case $V = \dfrac{102 + 5\, s'_{\overline{5|}}}{1 + \cdot 04\, s'_{\overline{5|}}}$ calculated at 3%

$$= \frac{102 + 5 \times 5{\cdot}3091}{1 + {\cdot}04 \times 5{\cdot}3091}$$

$$= 106{\cdot}029$$

The premium on the redemption value is thus 4·029, and the annual sinking fund instalment necessary to replace this is 4·029 ÷ 5·3091 = £·759. The balance of the yearly dividend which is available for interest is therefore £4·241, or exactly 4% of the purchase price of £106·029, thus showing the correctness of the original valuation.

In all cases where a separate sinking fund is actually set up to extinguish the premium on the redemption value, the ledger value of the security after each payment of dividend would be calculated by deducting the actual amount of the sinking fund accumulations from the original purchase price.

It is interesting to apply the method of this paragraph to the case where it is assumed that the sinking fund *can* be invested at the same rate of interest as that yielded by the security itself. In this case $i' = i$, and

$$V =$$

$$1 + \{(1 + i)^n - 1$$

$$R \,.\, v^n + D \,.\, a_{\overline{n|}}$$

which agrees exactly with formula (35) if $p = 1$.

11. So far, the valuation of the security has been undertaken as on a date immediately following a dividend payment. In practice, of course, valuations are frequently required on other dates, and in such cases, the simplest method is to calculate the value of the security on the last previous due-date of a dividend, and to add interest to the result, calculated for the period which has elapsed since that date. It would usually be quite sufficient to assume simple interest for this

period, and the value as on the last previous dividend due-date would accordingly be multiplied by a factor of the type $(1 + t \times i)$, where t represents the proportion of a year which has elapsed since the last previous dividend due-date, and i represents the effective rate of interest at which the valuation is being made. If the valuation were being made at a nominal rate of interest of j, convertible p times a year, the factor would be $(1 + t \times j)$, the frequency of conversion making no difference, since simple interest only is involved. On the assumption of simple interest, therefore, the factor is unity plus the product of the proportion of a *year* which has elapsed, and the *yearly* valuation rate of interest per unit. On the basis of compound interest, the above factors would be respectively $(1 + i)^t$ and $\left(1 + \frac{j}{p}\right)^{tp}$, but it is unusual in practice to use these compound interest factors, although it is theoretically correct to do so.

Suppose, for example, that it is desired to find the value on 5th July at a nominal rate of interest of 5% per annum convertible half-yearly of a debenture of £100, redeemable at £102 at the expiration of $7\frac{1}{2}$ years from the previous 30th April, and bearing interest at the rate of 4% per annum, payable half-yearly on 30th April and 30th October.

The value of the security on 30th April, when it had $7\frac{1}{2}$ years to run, was £102 $\times v^{15}$ + £2 $\times a_{\overline{15}|}$ calculated at $2\frac{1}{2}$%, or £95·191. Now from 30th April, to the following 5th July is 66 days, or ·1808 of a year, and the simple interest factor for accumulating a unit of capital at 5% per annum for this period is $(1 + \cdot 1808 \times \cdot 05) = 1 \cdot 00904$. So that the value of the security as on the given 5th July is £95·191 $\times$ 1·00904 = £96·051, or £96 1s. 0d. approximately.

EXERCISE IV

In answering these Questions, the tables at the end of the book may be used.

1. Find the present value of the following debentures, of which the nominal value is in each case £100—

	(a)	(b)	(c)
Redemption value .	£102	£103	£106
Unexpired term .	5 years	8 years	13 years
Yearly rate of interest payable . .	4% payable half-yearly	5% payable half-yearly	5% payable quarterly

2. What should be the ledger values of debenture (*a*) at the end of successive years ?

3. A debenture of nominal value £100, bearing interest at the rate of $4\frac{1}{2}\%$ per annum, payable half-yearly, is redeemable at £101 in 5 years, and is purchased for £103 1s. 4d., at which price it yields a rate of interest of 4% per annum, convertible half-yearly. It has been decided to set up a sinking fund, to accumulate at a rate of interest of 4% per annum, convertible half-yearly, to replace the premium on the redemption value. Calculate the half-yearly sinking fund payment.

4. A debenture of nominal value £100, bearing interest at the rate of $4\frac{1}{2}\%$ per annum, payable yearly, and redeemable at £102 in 15 years, is purchased to yield an effective rate of interest of 5% per annum. At what figure should it stand in the investment ledger at the expiration of 5 years ? Show arithmetically that if it were then sold at that figure, the original purchaser would have realized a rate of interest at 5% per annum for the period during which he held the security.

5. What price could a purchaser afford to pay for the debenture described below, if he required from his investment a nominal yield of 5% per annum, payable half-yearly, whereas he could set up a sinking fund to replace the premium on the redemption value at only 4% per annum, convertible half-yearly ?

Nominal value	£100
Redemption value . . .	£104
Unexpired term . . .	10 years
Yearly rate of interest payable .	$5\frac{1}{2}\%$ payable half-yearly

6. How would the yield be affected in the case of the last question if the purchaser paid a price of £107 ?

7. A firm has decided to issue debentures of nominal value £100 each, redeemable in 15 years, and bearing interest in the meantime at $4\frac{1}{2}\%$ per annum, payable half-yearly. The issue price is to be £98, and it is desired to fix the redemption value at such a figure that a purchaser would obtain an effective rate of interest of 5% per annum from the investment. Calculate the redemption value.

8. Find the present value as on 18th November, 1924, of the following debenture on the basis of a nominal rate of interest of 6% per annum, convertible half-yearly.

Nominal value	£100
Redemption value . . .	£102
Date of redemption . . .	1st July, 1939
Yearly rate of interest payable .	4% per annum, payable half-yearly on 1st Jan. and 1st July

CHAPTER V

LOANS REDEEMABLE BY INSTALMENTS, INCLUDING PRINCIPAL AND INTEREST

1. A PURCHASER of an immediate annuity-certain of 1 per annum payable yearly for n years, who desired to realize an effective rate of interest of i on his investment, would be willing to pay $a_{\overline{n}|}$ calculated at an effective rate of interest i.

The capital value of the investment at the outset would be $a_{\overline{n}|}$. By the end of the first year this capital value would accumulate to $(1 + i)\, a_{\overline{n}|} = \text{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$. But at the end of the first year the investor would receive the first annuity payment of unity; hence the accumulated capital value of $1 + a_{\overline{n-1}|}$ must then be reduced by unity, leaving a capital value of $a_{\overline{n-1}|}$ at the end of the first year.

By the end of the second year the capital value would have increased to $(1 + i)\, a_{\overline{n-1}|} = 1 + a_{\overline{n-2}|}$, which, after deduction of the annuity payment then due, would be reduced to $a_{\overline{n-2}|}$. And so the reasoning could be continued until it was shown that the capital value at the end of $(n-1)$ years would be $a_{\overline{n-(n-1)}|} = a_{\overline{1}|} = v$, the amount of which at the expiration of a further year would be unity, and equal to the final annuity payment then due.

It is seen, therefore, that the purchaser of the annuity would secure, firstly, the return of his principal (since none would remain outstanding at the end of the period), and, secondly, yearly interest at the effective rate i, calculated on so much of his principal as remained outstanding at the beginning of successive years.

2. Instead of regarding the transaction as a sale and purchase of an annuity-certain, it may be considered as a loan of $a_{\overline{n}|}$ granted by the purchaser to the vendor, to be liquidated by n yearly instalments of 1 per annum, such instalments including both principal and interest.

As has already been shown, the capital value of the investment gradually diminishes from $a_{\overline{n}|}$ until it finally vanishes by the end of the period of the loan. The annual interest on the successive amounts of principal outstanding must therefore correspondingly decrease, and, as the total yearly

instalment (which includes both principal and interest) remains the same throughout the period, the amount of principal included in the instalments must increase from year to year.

Before showing algebraically how the respective instalments may be analysed into principal and interest, it will, perhaps, be advisable to give an arithmetical illustration of the various points already mentioned.

3. Let it be assumed that an intending borrower is willing to pay £1 per annum for 3 years in consideration of a present advance, and that a lender can be found, who would be agreeable to make the advance on the basis of an effective rate of interest of 4% per annum.

On this basis, the lender would advance a sum of £$a_{\overline{3}|}$, calculated at a rate of interest of 4% per annum, or £2·7751.

The following statement shows how the transaction would work out—

	£
Amount of advance	2·7751
Add first year's interest thereon at 4%	·1110
	2·8861
Deduct first instalment	1·0000
Principal outstanding at end of 1st year	1·8861
Add second year's interest thereon at 4%	·0754
	1·9615
Deduct second instalment	1·0000
Principal outstanding at end of 2nd year	·9615
Add third year's interest thereon at 4%	·0385
	1·0000
Deduct third instalment	1·0000
Principal outstanding at end of 3rd year	Nil

It will be seen—

(*a*) That the whole of the advance is repaid by the end of the term;

(*b*) that the lender receives interest at the rate of 4% per annum on so much of his principal as remains outstanding at the beginning of each year; and

(*c*) that (on referring to compound interest tables) the principal outstanding at the end of any year, after payment of the instalment then due, is the "present value" of the remaining instalments (see paragraph 1).

4. The scheme of repayment may, however, be considered from a slightly different point of view. At the end of the first year a sum of 4% of £2·7751, i.e. £·1110, becomes due for interest; at the same time the borrower pays the first

instalment of £1. £·1110 of this instalment thus being due for interest, the balance of £·8890 is applied in reducing the amount of the principal outstanding from £2·7751 to £1·8861. Similarly, at the end of the second year, interest of 4% of £1·8861, i.e. £·0754, is due, and, after deduction of this amount from the second instalment of £1, there remains a balance of £·9246, which reduces the principal outstanding to £·9615 At the end of the third year, interest of 4% of £·9615, i.e. £·0385, becomes due, and this amount is deducted from the final instalment of £1, leaving a balance of £·9615 to repay the whole of the principal then remaining outstanding.

These results may be conveniently set out in schedule form as follows—

Year No.	Principal outstanding at beginning of year.	Interest for year.	Principal contained in instalment for year.
Col. (1)	Col. (2)	Col. (3)	Col. (4)
	£	£	£
1	2·7751	·1110	·8890
2	1·8861	·0754	·9246
3	·9615	·0385	·9615
		TOTAL	£2·7751

5. In the case of a loan of $a_{\overline{n}|}$, granted at an effective rate of interest of i, and repayable in n equal annual instalments of 1, including principal and interest, the schedule would be as follows—

Year No.	Principal outstanding at beginning of year.	Interest for year.	Principal contained in instalment for year.
Col. (1)	Col. (2)	Col. (3)	Col. (4)
1	$a_{\overline{n}|}$	$1-v^n$	v^n
2	$a_{\overline{n-1}|}$	$1-v^{n-1}$	v^{n-1}
3	$a_{\overline{n-2}|}$	$1-v^{n-2}$	v^{n-2}
—	—	—	—
—	—	—	—
$n-2$	$a_{\overline{3}|}$	$1-v^3$	v^3
$n-1$	$a_{\overline{2}|}$	$1-v^2$	v^2
n	$a_{\overline{1}|}$	$1-v$	v
		TOTAL	$a_{\overline{n}|}$

The method of constructing this schedule is the same as in the case of the above arithmetical example. At the risk, however, of a certain amount of repetition, the working will be described.

The schedule is commenced by inserting $a_{\overline{n}|}$ as the principal outstanding at the beginning of the first year. The interest for the first year is $i \times a_{\overline{n}|} = 1 - v^n$. The instalment due from the borrower at the end of the year is 1, and, as $1 - v^n$ of this sum is due for interest, the balance of v^n is available to reduce the amount of the loan. Accordingly, v^n is entered as the principal contained in the instalment for the first year. The principal outstanding immediately after the first instalment has been paid will therefore be

$$\begin{aligned} a_{\overline{n}|} - v^n &= (v + v^2 + v^3 + \ldots + v^{n-1} + v^n) - v^n \\ &= v + v^2 + v^3 + \ldots + v^{n-1} \\ &= a_{\overline{n-1}|}, \text{ which is entered in column (2).} \end{aligned}$$

The cycle of operations is then recommenced, the interest in column (3) being $i \times a_{\overline{n-1}|} = 1 - v^{n-1}$, the principal contained in the second instalment being v^{n-1}, and so on.

6. It is to be observed that the quantities in column (4) of the schedule increase in geometric progression, the common ratio being $(1 + i)$ and the first term being v^n. If this relationship is kept in mind, it will frequently assist in the prevention of arithmetical errors in the construction of a schedule, or in the detection of an error which may already have escaped notice. The relationship also furnishes a convenient alternative means of constructing a schedule if a multiplying machine is available ; the initial figure in column (4) would first be calculated, and the remaining figures in the column would be obtained by multiplying successively by $(1 + i)$.

7. It has been assumed so far that the annual instalment is unity. If, however, the loan and the rate of interest have been fixed, and it is desired to know what the annual instalment should be, the problem may be solved by simple proportion. For example, supposing that the loan is to be repaid by n annual instalments, the instalment would be calculated as follows—

A loan of $a_{\overline{n}|}$ will be repaid by an annual instalment of 1.

Therefore a loan of 1 will be repaid by an annual instalment of $\frac{1}{—}$.

Whence a loan of L will be repaid by an annual instalment of $L \times \dfrac{1}{a_{\overline{n}|}}$.

The values of $\dfrac{1}{a_{\overline{n}|}}$ are sometimes tabulated in compound interest tables, but, in their absence, the amount of the loan would have to be divided by $a_{\overline{n}|}$ to find the annual instalment in any given instance.

8. In all cases the construction of a schedule is based on the principles which have already been described, but in practice the results would have to be expressed in pounds, shillings and pence, after the calculations had been made to 4 places of decimals. In all probability certain minor adjustments would have to be made to the pence figures, so that the total of column (4) should be exactly equal to the sum advanced. In order to ensure consistency, it is suggested that the best method of procedure is first to convert the decimal fractions of a £ in this column to their sterling equivalents (to the nearest penny); then to find the total of the column, and finally to make small adjustments (not exceeding one penny in each case) so as to correct any error in the total. In making these adjustments, care must be taken not to sacrifice unnecessarily the accuracy of any individual item in the column as judged by the original calculations.

Having completed column (4), the sterling figures appropriate to column (3) should be calculated by deducting the corresponding column (4) figures from the annual instalment, expressed to the nearest penny. Lastly, column (2) should be completed by continued deduction of the respective column (4) items, as was done when the schedule was originally constructed.

Whatever method of adjustment is adopted, however, neither the conversion of the decimals of a £ into shillings and pence, nor any adjustment of the sterling amounts should be undertaken without tracing the effect upon the remaining figures in the schedule.

9. If the loan were to be liquidated by instalments payable more frequently than yearly, it would be assumed in practice that the rate of interest was convertible with the same frequency. The key to the solution of any problem in such a case is to change the unit of time-measurement from one year to the period which is to elapse between successive

instalments. For instance, if a loan of £150 bearing interest at 5% per annum is to be repaid in 5 years by half-yearly instalments, including principal and interest, the amount of the half-yearly instalment would be taken as $\frac{£150}{a_{\overline{10}|}}$ calculated at a rate of interest of $2\frac{1}{2}\%$ = £17·1389 = £17 2s. 9d. to the nearest penny.

In this case, the preliminary calculation of the schedule to 4 places of decimals would stand as follows—

Half-year No.	Principal outstanding at beginning of half-year.	Interest for half-year.	Principal contained in instalment for half-year.
Col. (1)	Col. (2)	Col. (3)	Col. (4)
1	150·0000	3·7500	13·3889
2	136·6111	3·4153	13·7236
3	122·8875	3·0722	14·0667
4	108·8208	2·7205	14·4184
5	94·4024	2·3601	14·7788
6	79·6236	1·9906	15·1483
7	64·4753	1·6119	15·5270
8	48·9483	1·2237	15·9152
9	33·0331	·8258	16·3131
10	16·7200	·4180	16·7209
		TOTAL	150·0009 *

After adjustment on the lines recommended in the last paragraph, the schedule would finally be as follows—

Half-year No.	Principal outstanding at beginning of half-year.	Interest for half-year.	Principal contained in instalment for half-year.
Col. (1)	Col. (2)	Col. (3)	Col. (4)
	£ s. d.	£ s. d.	£ s. d.
1	150 – –	3 15 –	13 7 9
2	136 12 3	3 8 3	13 14 6
3	122 17 9	3 1 5	14 1 4
4	108 16 5	2 14 5	14 8 4
5	94 8 1	2 7 2	14 15 7
6	79 12 6	1 19 9	15 3 –
7	64 9 6	1 12 2	15 10 7
8	48 18 11	1 4 6	15 18 3
9	33 – 8	– 16 6	16 6 3
10	16 14 5	– 8 4	16 14 5
		TOTAL	£150 – –

* The slight discrepancy in the total is due to the limitation of the number of decimal places in the calculations.

10. In practice, loans of the type discussed in this chapter are made in connection with mortgages of leasehold property and other "wasting" securities of which the value will necessarily depreciate as time goes on. When an advance is required on such a security, it is only prudent for the lender to stipulate for a scheme of repayment which will ensure that the principal outstanding at any time will be within the value of the security at that time.

In connexion with these loans, it is customary to include in the mortgage deed a schedule of the above kind showing how the redemption of the loan is to be effected. This schedule would serve the practical purpose of showing the amount of interest which is included in each instalment, and upon which income tax would be payable by the lender. Also, if the borrower had the right to redeem at any time, the schedule would indicate the sum which would be required to repay the advance.

11. It has hitherto been assumed that the loan will successively be reduced by so much of each instalment as will not be required for interest. Alternatively, however, interest on the *whole* of the original principal may be taken from the instalment on each occasion, and the balance left to be invested as a sinking fund payment so as to redeem the loan in its entirety by the end of the term. The principal, which, under the former mode of treatment, would be shown as outstanding at the beginning of any interval would then be represented by the difference between the sum originally advanced and the sinking fund accumulations to date.

For instance, in the example of paragraph 9, each half-yearly instalment of £17·1389 could be regarded as consisting of (*a*) the half-yearly interest of £3·75 on £150, and (*b*) a half-yearly sinking fund payment of £13·3889. The accumulation of these sinking fund payments is shown in the statement on page 54; the figures in column (2) of this statement could, of course, have been directly obtained by multiplying the half-yearly sinking fund payment by the $2\frac{1}{2}\%$ values of s from $s_{\overline{1}|}$ to $s_{\overline{10}|}$ inclusive, but it is thought that the detailed illustration of the actual process of accumulation will prove instructive.

The figures in column (5) of the statement will be seen to agree with those in column (2) of the schedule in paragraph 9.

12. To demonstrate the last paragraph algebraically, take the case of the loan of $a_{\overline{n}|}$ referred to in paragraph 5. Each annual instalment of 1 may be taken to consist of

(*a*) Interest at the effective rate i on the entire advance of $a_{\overline{n}|}$, viz. $(1 - v^n)$; and

(*b*) Annual sinking fund payment of v^n.

The amount of the sinking fund accumulations by the end of the term would be $v^n \times s_{\overline{n}|}$ (calculated at an effective rate of interest i) $= a_{\overline{n}|}$, the sum originally advanced.

And the amount of the sinking fund accumulations by the end of m years (m being less than n) would be $v^n \times s_{\overline{m}|}$. The difference between the sum originally advanced and these accumulations would be

$$a_{\overline{n}|} - v^n \times s_{\overline{m}|}$$

$$i \qquad\qquad i$$

$$. - v^n \qquad v^{n-m} - v^n$$

$$1 - v^{n-n}$$

which would be the principal outstanding immediately after the m^{th} instalment had been paid, if the instalments were analysed as in paragraph 5.

Statement showing the accumulation of a half-yearly sinking fund payment of £13·3889 at a nominal rate of interest of 5% per annum, convertible half-yearly.

Half-year No.	Total amount of sinking fund accumulations at end of half-year.	Interest thereon earned in next half-year.	Half-yearly sinking fund payment due at end of next half-year.	Original loan less sinking fund accumulations stated in Col. (2).
Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
1	£13·3889	£·3347	£13·3889	£136·6111
2	27·1125	·6778	13·3889	122·8875
3	41·1792	1·0295	13·3889	108·8208
4	55·5976	1·3899	13·3889	94·4024
5	70·3764	1·7594	13·3889	79·6236
6	85·5247	2·1381	13·3889	64·4753
7	101·0517	2·5263	13·3889	48·9483
8	116·9669	2·9242	13·3889	33·0331
9	133·2800	3·3320	13·3889	16·7200
10	150·0009	—	—	Loan repaid

13. It has been shown in the last paragraph that where a loan of $a_{\overline{n}|}$, which has been granted at an effective rate of interest i, is being liquidated by annual instalments of unity, including principal and interest, each instalment may be analysed into annual interest of $i \times a_{\overline{n}|}$, and annual sinking fund payment of v^n. By simple proportion, therefore, each *unit* of loan requires an annual instalment of $\frac{1}{a_{\overline{n}|}}$ consisting of (a) annual interest of i, and (b) annual sinking fund payment of $\frac{v^n}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}}$. It is, moreover, evident without demonstration that the yearly interest on an advance of 1 is i, and that the annual sinking fund payment necessary to replace a sum of 1 at the end of n years is $\frac{1}{s_{\overline{n}|}}$

Therefore

A direct algebraical proof of this relationship is as follows—

$$= i + \frac{iv^n}{1 - v^n}, \text{ by actual division,}$$

on multiplying the numerator and denominator of the fraction by $(1 + i)^n$

It may be added, incidentally, that the values of $\frac{1}{s_{\overline{n}|}}$ are frequently required in the calculation of sinking fund payments which arise in other connexions, but in view of the simple relationship which is seen to exist between $\frac{1}{a_{\overline{n}|}}$ and $\frac{1}{{}_{\overline{n}|}}$,

it is not usual to give the values of both of these functions in compound interest tables.

14. It has been shown that if the lender does not actually set up a sinking fund, he will secure the stipulated rate of interest on so much only of the principal as remains outstanding at the beginning of each interval. If, however, he does set up a sinking fund, he will secure that rate of interest on the whole of the original amount of the advance, provided, however, that he can reinvest the sinking fund payments and the interest thereon at the same rate of interest as that charged on the loan. The reservation is most important, because, if the sinking fund payments were invested at a smaller rate of interest, they would not suffice to repay the original principal by the end of the term.

If, therefore, at the outset the lender required to earn a certain rate of interest on the *whole* of his capital for the entire period of the loan, he would have to consider whether he would be able to reinvest the sinking fund payments at the same rate.

It is quite probable that he would be able to reinvest at only a lower rate, and the question thus arises as to how the repayment of the loan should be effected if such were the circumstances.

Let i' = the effective rate of interest per annum—usually described as the "remunerative" rate—which the lender desires to receive on the whole of his capital for the entire duration of the advance.

And let i = the effective rate of interest per annum—commonly described as the "reproductive" rate—at which the sinking fund payments are to be invested.

As regards these two rates of interest, the problem is simplified if the reproductive rate is regarded as the general market rate of interest at which all sums coming into the possession of the lender will be invested, whereas the remunerative rate of interest is peculiar to the particular transaction in hand.

Now, for each unit of loan advanced, there will be required i' for the annual interest, and $\frac{1}{s_{\overline{n}|}}$, calculated at rate i, for

the annual sinking fund payment, so that the yearly instalment necessary to repay the loan will be $i' + \frac{1}{s_{\overline{n}|}}$

Since $\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i$, this annual instalment may be alternatively expressed as $\frac{1}{a_{\overline{n}|}} + (i' - i)$, which shows that the instalment may be calculated in these cases by adding extra annual interest at the rate $(i' - i)$ on the *whole* advance to the annual instalment computed on the basis of the reproductive rate alone.

15. Two rates of interest may, however, arise in the same transaction at the instance of the borrower, and not of the lender. Suppose, for example, that a borrower raises a loan of £10,000 at 5% interest to be repaid in one sum at the end of 10 years, interest of £500 per annum being paid by him throughout this period. He might think it advisable to create a sinking fund on his own account, and so to avoid the possible inconvenience of having to find so large a sum as £10,000 in one amount at the expiry of the loan. If he were able to invest at only 4% per annum, the annual sinking fund instalment which he would have to set aside would be $\frac{£10{,}000}{s_{\overline{10}|}}$, calculated at 4%, = £832·91. The total annual sum required for the "service" of the loan would thus be £1,332·91, inclusive of the annual interest of £500.

16. As an illustration of the working of a loan when remunerative and reproductive rates of interest are involved, consider the following example, which is similar to that given in paragraph 9. A loan of £150 is to be repaid in 5 years by half-yearly instalments, including principal and interest; the lender requires to earn a rate of interest of 6% per annum, payable half-yearly, on the whole of the loan, but can reinvest the sinking fund at only 5% per annum, convertible half-yearly.

The half-yearly instalment would consist of—

(*a*) Interest of 3% on £150 = £4·5, and

(*b*) Sinking fund of £150 $\div s_{\overline{10}|}$, calculated at $2\frac{1}{2}\% =$ £13·3889.

The half-yearly instalment would thus be £17·8889, or $\frac{1}{2}$% of £150 more than that in paragraph 9. (See end of paragraph 14.)

Particulars of the loan and the sinking fund accumulations may be given in schedule form as follows—

Half-year No.	Original Loan.	Interest thereon for half-year.	Total amount of Sinking Fund accumulations at end of half-year.	Interest thereon earned in next half-year.
Col. (1)	Col. (2) £	Col. (3) £	Col. (4) £	Col. (5) £
1	150	4·5	13·3889	·3347
2	150	4·5	27·1125	·6778
3	150	4·5	41·1792	1·0295
4	150	4·5	55·5976	1·3899
5	150	4·5	70·3764	1·7594
6	150	4·5	85·5247	2·1381
7	150	4·5	101·0517	2·5263
8	150	4·5	116·9669	2·9242
9	150	4·5	133·2800	3·3320
10	150	4·5	150·0009	—

17. These results may be compressed into the more usual type of schedule as follows, although the latter hardly applies in these cases.

Half-year No.	Principal out-standing at beginning of half-year.	Interest for half-year.	Principal contained in instalment for half-year.
Col. (1)	Col. (2) £	Col. (3) £	Col. (4) £
1	150·0000	4·5000	13·3889
2	136·6111	4·1653	13·7236
3	122·8875	3·8222	14·0667
4	108·8208	3·4705	14·4184
5	94·4024	3·1101	14·7788
6	79·6236	2·7406	15·1483
7	64·4753	2·3619	15·5270
8	48·9483	1·9737	15·9152
9	33·0331	1·5758	16·3131
10	16·7200	1·1680	16·7209

The figures in column (2) represent the result of deducting the sinking fund accumulations from the original loan. The principal contained in each instalment, as stated in column (4), represents the reduction which is made in the principal outstanding immediately after the relative instalment has been paid. The interest stated in column (3) represents the

balance of the instalments after deducting the principal contained therein, as stated in column (4).

If this latter schedule be now compared with that in paragraph 9, it will be seen that the only difference is that the interest for each half-year has been increased by £·7500, i.e. by the additional interest which is required by the lender. The formula at the end of paragraph 14 shows that this relationship holds in all cases where remunerative and reproductive rates of interest enter the calculations. In practice, therefore, a convenient method of constructing these schedules is first to work on the basis of the reproductive rate of interest alone, and then to increase the interest column by the extra periodic interest on the *whole* of the original advance.

EXERCISE V

In answering these Questions the tables at the end of the book may be used.

1. A loan of £125 has been granted at a rate of interest of 5% per annum, and is repayable in 5 equal annual instalments, including principal and interest. Find the amount of the annual instalment.

2. Construct a schedule for the above loan, showing the analysis of each instalment into principal and interest, and the amount of principal outstanding after each instalment is paid.

3. A loan of £250 is to be liquidated by 6 equal half-yearly instalments, including principal and interest at the rate of 6% per annum. Calculate the amount of the half-yearly instalment, and draw up a schedule showing the repayment of the loan.

4. A loan of £150 is to be repaid by means of 6 equal annual instalments, including principal and interest. The lender requires to realize 5% per annum on the full amount of the loan for the whole period of 6 years, but can reinvest the necessary sinking fund at only 4½% per annum. Calculate the amount of each yearly instalment, and draw up a schedule for insertion in the mortgage deed for income tax purposes.

5. In the case of the last question, find the amount of the sinking fund accumulations immediately after the fourth instalment has been paid.

6. A loan of £235 has been granted at a rate of interest at 6% per annum, and is being repaid by means of 40 equal quarterly instalments, including principal and interest. Find, without constructing a schedule (*a*) the amount of interest contained in the eleventh quarterly instalment, and (*b*) the principal outstanding immediately after that instalment has been paid.

7. A building society grants loans on the security of house property at the rate of 5% per annum, such loans being repayable by equal monthly instalments, including principal and interest. Each borrower's account is made up by the society yearly, and, although the repayments are actually made monthly, interest is charged upon the full amount of the principal outstanding at the beginning of the year of account. Calculate the monthly instalment which would be required in these circumstances to repay a loan of £500 by the end of 12 years.

CHAPTER VI

SINKING FUND ASSURANCES

1. A PERSON who desires to set up a sinking fund in respect of any type of wasting security, or to replace the capital loss arising from depreciation of machinery, etc., usually finds that he is faced with two practical difficulties. The first is the uncertainty of being able to find suitable investments from time to time, which will yield the desired rate of interest (after deduction of income tax), and will be realizable at their full book value by the time the sinking fund is to terminate. The second is the difficulty, amounting in practice almost to an impossibility, of promptly investing each amount of dividend as soon as it is paid, on the same terms as the sinking fund payments themselves; this difficulty is, of course, increased if the dividends are small in actual amount. The operation of the sinking fund might, therefore, require a closer attention to investment matters than it was convenient to give, and would inevitably be attended by the usual investment expenses, such as broker's commission, stamp duty, etc., and, probably, by the expenses incidental to the clerical work of supervising and selecting the investments.

To meet these difficulties, insurance companies are prepared to issue policies which provide for the payment of a capital sum on a fixed date in consideration of the payment to them of a single amount when the policy is effected, or, alternatively, in consideration of the payment of a periodic amount.

These contracts are usually described as sinking fund assurances, or capital redemption assurances, but they are occasionally called leasehold assurances, owing to their being frequently effected to replace the loss of capital which occurs on the expiry of the term of leasehold property.

2. The capital sum secured by a sinking fund policy is known as the sum assured, and the date upon which it is payable is described as the date of maturity. The consideration paid by the person effecting the policy is called the premium. Premiums may be paid in one sum at the outset, in which case they are called single premiums, or they may be paid periodically until the policy matures. These periodic premiums are customarily paid annually, half-yearly, or quarterly, and it is important to note that they are payable

in advance, the first being due at the outset, and the last being due one period before the date of maturity.

3. The single premium under a sinking fund policy is evidently such a sum as will accumulate at interest to the sum assured by maturity. The single premium necessary to secure a sinking fund policy for a sum assured of 1 payable at the expiration of n years is symbolically denoted by $A_{\overline{n}|}$.

Therefore, at an effective rate of interest i,

$$A_{\overline{n}|} \times (1+i)^n = 1$$

Whence $$A_{\overline{n}|} = \frac{1}{(1+i)^n} \text{ or } v^n \quad . \quad . \quad . \quad (39)$$

It is clear from general considerations that the single premium must represent the present value of the sum assured, and $A_{\overline{n}|}$ could therefore have been written down straight away as v^n.

The rate of interest which would be employed by the insurance company in calculating premiums would naturally be somewhat below the net rate (after deduction of income tax) which it would expect to earn in the future. The excess of the net rate of interest actually realized over the rate at which the premiums are calculated would constitute the source from which the expenses of conducting the business would be met, and the profits would be earned.

4. The annual premium under a sinking fund assurance may be found by equating the accumulated amount of the premiums payable during the term to the sum assured. The symbol $P_{\overline{n}|}$ is customarily employed to denote the annual premium to secure a sum assured of unity payable at the expiration of n years from the original date of assurance.

Thus $$P_{\overline{n}|}\,(s_{\overline{n+1}|} - 1) = 1$$

Whence $$P_{\overline{n}|} = \frac{1}{s_{\overline{n+1}|} - 1} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$P_{\overline{n}|}$ differs in form and value from the corresponding sinking fund payment of $\frac{1}{s_{\overline{n}|}}$ dealt with in the previous chapter, because the annual premiums of $P_{\overline{n}|}$ are payable at the *beginning* of each year (and thus constitute annuities-due) whereas the sinking fund payments of $\frac{1}{s_{\overline{n}|}}$ are payable at the *end* of each

year (and thus constitute immediate annuities). The relationship subsisting between the two types of sinking fund payment is

which may be easily established either by general reasoning or by algebraic methods.

5. Alternatively, the annual premium may be found by equating the present values of the sum assured and of all the annual premiums.

Thus $$P_{\overline{n}|} \times a_{\overline{n}|} = v^n$$

Whence $$P_{\overline{n}|} = \frac{v^n}{a_{\overline{n}|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

It may be shown that $\frac{v^n}{a_{\overline{n}|}}$ is the same as $\frac{1}{s_{\overline{n+1}|} - 1}$, by multiplying the numerator and denominator of the former expression by $(1 + i)^n$.

It is interesting to observe that the following relationship exists between the single and annual premiums, viz.

$$P_{\overline{n}|} \times a_{\overline{n}|} = A_{\overline{n}|}$$

which indicates that the single premium is the exact equivalent of the present value of all the annual premiums payable.

6. As an illustration of the application of formulas (39) and (40), the single and annual premiums will be found for a sinking fund assurance for £100 maturing at the end of 20 years, the calculations being based on an effective rate of interest of 3% per annum.

The desired single premium $= £100 \times v^{20}$
$= £55{\cdot}368$ or £55 7s. 4d.

The desired annual premium $= £100 \div (s_{\overline{21}|} - 1)$
$= £100 \div 27{\cdot}6765$
$= £3{\cdot}613$ or £3 12s. 3d.

Premiums Payable more Frequently than Annually

7. Premiums payable more frequently than annually may be calculated on the same principles as those described in paragraphs 4 and 5.

Let $P^{(m)}_{\overline{n}|}$ denote the total of such premiums payable in each *year* in equal instalments of $\frac{P^{(m)}_{\overline{n}|}}{m}$ at the beginning of each $\frac{1}{m}$th of a year to secure a sum assured of 1 at the end of n years.

Then, working on the basis of the equality of the accumulated amount of the premiums to the sum assured, as in paragraph 4,

$$P^{(m)}_{\overline{n}|} \times (1 + i)^{\frac{1}{m}} \times s^{(m)}_{\overline{n}|} = 1 \quad \text{(See formula (31))}$$

$$\text{Whence } P^{(m)}_{\overline{n}|} = \frac{1}{(1 + i)^{\frac{1}{m}} \times s^{(m)}_{\overline{n}|}} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Or, working on the basis of the equality of the present values of the sum assured and the premiums as in paragraph 5,

$$P^{(m)}_{\overline{n}|} \times a^{(m)}_{\overline{n}|} = v^n$$

$$\text{Whence } P^{(m)}_{\overline{n}|} = \frac{v^n}{a^{(m)}_{\overline{n}|}}$$

$$= \frac{v^n}{(1 + i)^{\frac{1}{m}} \times a^{(m)}_{\overline{n}|}} \quad \text{(See formula (29))} . \quad (43)$$

The equality of the two forms for $P^{(m)}_{\overline{n}|}$, as shown by formulas (42) and (43), will be evident if it is remembered that

$$a^{(m)}_{\overline{n}|} = v^n \times s^{(m)}_{\overline{n}|}.$$

8. As an illustration of the calculation of sinking fund premiums payable more frequently than yearly, suppose that the half-yearly and quarterly premiums had been required in the case of the example in paragraph 6, instead of the annual premium.

Formula (42) would be the more convenient to apply, the desired half-yearly premium being given by

$$£100 \times \frac{1}{2} P^{(2)}_{\overline{20}|} = \frac{1}{2} \times \frac{£100}{(1 + i)^{\frac{1}{2}} \times \frac{i}{j} \times s_{\overline{20}|}} \text{ calculated at } 3\%$$

$$= \frac{1}{2} \times \frac{£100}{1{\cdot}01489 \times 1{\cdot}00744 \times 26{\cdot}8704}$$

$$= £1{\cdot}820 \text{ or } £1 \ 16\text{s. } 5\text{d.}$$

Similarly, the desired quarterly premium would be given by

$$\pounds 100 \times \frac{1}{4} P^{(4)}_{\overline{20}|} = \frac{1}{4} \times \frac{\pounds 100}{\text{[illegible]}} \text{calculated at } 3\%$$

$$= \frac{1}{4} \times \frac{\pounds 100}{1{\cdot}00742 \times 1{\cdot}01118 \times 26{\cdot}8704}$$

$$= \pounds{\cdot}913 \text{ or } 18s.\ 3d.$$

It will have been observed that the premiums per unit of the sum assured have been taken as one-half and one-quarter of the respective values of $P^{(2)}_{\overline{20}|}$ and $P^{(4)}_{\overline{20}|}$; this is because $P^{(m)}_{\overline{n}|}$ gives the total of all the $\frac{1}{m}$thly premiums payable in a year, the $\frac{1}{m}$thly premium itself being $\frac{P^{(m)}_{\overline{n}|}}{m}$

9. The value of $P^{(m)}_{\overline{n}|}$ is always greater than that of $P_{\overline{n}|}$ because the payment of premiums in instalments instead of in advance at the beginning of each year entails a certain loss of interest to the insurance company, which loss has to be made good by charging a higher rate of premium.

A fixed relationship always exists, however, between $P^{(m)}_{\overline{n}|}$ and $P_{\overline{n}|}$ because the instalment premiums falling due in each year are the exact equivalent of an annual premium if due allowance be made for interest.

Thus $$P^{(m)}_{\overline{n}|} \times \ddot{a}^{(m)}_{\overline{1}|} = P_{\overline{n}|}$$

Whence $$P^{(m)}_{\overline{n}|} = \frac{P_{\overline{n}|}}{\ddot{a}^{(m)}_{\overline{1}|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

It will thus be seen that the values of $P^{(m)}_{\overline{n}|}$ may be obtained from those of $P_{\overline{n}|}$ by multiplying by the factor $\frac{1}{\ddot{a}^{(m)}_{\overline{1}|}}$, which is independent of n. The corresponding factor by which the annual premium must be multiplied to give the $\frac{1}{m}$thly premium itself is $\frac{1}{m} \times \frac{1}{\ddot{a}^{(m)}_{\overline{1}|}}$. To lend reality to the matter,

a few specimen values of this latter factor are given below. These have been calculated from the formula

$$\frac{1}{m \times a_{\overline{1|}}^{(m)}} \quad \frac{1 - v^{\frac{1}{m}}}{1 - v}$$

which may be easily derived if it is remembered that

$$\times$$

$$m\{(1$$

Effective Rate of Interest i.	Factor Applicable to	
	Half-yearly Premiums.	Quarterly Premiums.
·02	·50248	·25186
·025	·50309	·25232
·03	·50370	·25278
·035	·50430	·25323
·04	·50490	·25369

By the use of this table, the half-yearly and quarterly premiums calculated in paragraph 8 could be quickly found from the annual premium of £3 12s. 3d. Thus—

The half-yearly premium = £3·613 × ·50370
= £1·820 as before, and

The quarterly premium = £3·613 × ·25278
= £·913 as before.

Approximate values of the above factors may be obtained by assuming that the premiums in each year accumulate at *simple* interest. On this assumption, in the case of quarterly premiums, the first premium of $\frac{1}{4}P_{\overline{n|}}^{(4)}$ would earn a full year's interest, the second would earn three-quarter's of a year's interest, and so on, the total amount to which the premiums would have accumulated by the end of the year being

An annual premium, on the other hand, would have accumulated by the end of the year to

and since the quarterly premiums are the exact equivalent of an annual premium, these two expressions must be equal, i.e.

$$+ \tfrac{3}{8}i\} = P_{\overline{n}|} \{1 + i\}$$

$$P^{(4)} \qquad P \qquad 1 + i$$

$$= P_{\overline{n}|} (1 + \tfrac{3}{8}i) \text{ approximately, by actual}$$

division.

Thus the quarterly premium $= \frac{1}{4}(1 + \frac{3}{8}i) \times$ annual premium, approximately. Similarly, it may be shown that the half-yearly premium is approximately $\frac{1}{2}(1 + \frac{1}{4}i) \times$ annual premium.

Policy Values

10. The premiums received by an insurance company under its sinking fund contracts are carried to a separate fund, out of which the sums assured are paid as they become due, and the expenses of management and other incidental outgo are defrayed. All sums appropriate to this fund which are not immediately required to meet outgo would be invested by the company, and the net interest earned, after deduction of income tax, would be credited to the fund. The amount of the fund at any particular time would show the total sum which was held invested on behalf of the sinking fund policy-holders as a class, and the question naturally arises as to whether this amount would enable the respective sums assured to be paid by maturity, assuming, of course, that the payment of future premiums will be duly made by the policy-holders.

The amount which should be held by the company against any particular sinking fund policy is known as the reserve or policy value. These policy values may be calculated by two methods, viz. (1) the retrospective method, under which the policy value is represented as the accumulated amount of the premiums already paid; and (2) the prospective method, under which the policy value is represented as the excess of the present value of the sum assured over the present value of all of the future premiums. Regarded from the point of view of the prospective method, the policy value is clearly such a sum as will, with the future premiums, enable the sum

assured to be paid at maturity. If the retrospective method of calculation be employed, it is, of course, essential that the net rate of interest used in the accumulation of the premiums should not be less than the net rate upon which the premiums themselves were based; otherwise the policy value at the date of maturity would be less than the sum assured then due. Ordinarily, however, the calculations under both methods would be made at the same rate of interest as that adopted in the construction of the scale of premiums.

11. In the case of a single premium policy effected to secure a sum assured of unity at the end of n years by payment of a single premium of $\frac{1}{(1+i)^n}$, the policy value at the end of t years calculated according to the retrospective method would clearly be $\frac{1}{(1+i)^n} \times (1+i)^t$ or v^{n-t}.

As there are no further premiums payable, the policy value according to the prospective method would be simply the present value of the sum assured, viz. v^{n-t}.

In the case of an annual premium policy effected to secure a sum assured of unity at the end of n years, subject to payment of annual premiums of $P_{\overline{n|}}$, the policy value at the end of t years (immediately before payment of the premium then due) would be denoted by ${}_tV_{\overline{n|}}$. According to the retrospective method of calculation, therefore,

$$ {}_tV_{\overline{n|}} = P_{\overline{n|}}\,(s_{\overline{t+1|}} - 1) \qquad . \quad . \quad (45)$$

According to the prospective method, the present value of the sum assured being v^{n-t}, and the present value of the future annual premiums being $P_{\overline{n|}}\,a_{\overline{n-t|}}$,

$$ {}_tV_{\overline{n|}} = v^{n-t} - P_{\overline{n|}}\,a_{\overline{n-t|}} \qquad . \quad . \quad (46)$$

It is shown in the Appendix that the policy values according to the two methods are identical, provided that the same rate of interest is employed throughout.

12. As an illustration of the calculation of policy values, suppose that it is desired to find the respective policy values at the end of 8 years of the policies for which the single and annual premiums were obtained in paragraph 6.

The policy value at the end of 8 years of the single premium policy calculated according to the retrospective method would be

$$£55{\cdot}368 \times (1{\cdot}03)^8 = £55{\cdot}368 \times 1{\cdot}26677 = £70{\cdot}14.$$

According to the prospective method, this policy value would be

$$£100 \times v^{12} \text{ at } 3\% = £100 \times \cdot 70138 = £70{\cdot}14, \text{ as before.}$$

The policy value at the end of 8 years of the annual premium policy calculated according to the retrospective method would be

$$£3{\cdot}613 \times (s_{\overline{9}|} - 1) \text{ at } 3\% = £3{\cdot}613 \times 9{\cdot}1591 = £33{\cdot}09$$

According to the prospective method, this policy value would be

$$£100 \times v^{12} - £3{\cdot}613 \times a_{\overline{12}|} \text{ at } 3\%$$
$$= £70{\cdot}138 - £3{\cdot}613 \times 10{\cdot}2526 = £33{\cdot}09, \text{ as before.}$$

13. The policy values of sinking fund assurances effected by premiums payable m times a year will now be considered. The policy value at the end of t years, and immediately before the due-date of a premium, of a sinking fund assurance securing a sum assured of 1 payable at the end of n years and effected at $\frac{1}{m}$thly premiums is denoted symbolically by

According to the retrospective method,

$$_tV^{(m)}_{\overline{n}|} = P^{(m)}_{\overline{n}|} \times (1+i)^{\frac{1}{m}} \times s^{(m)}_{\overline{t}|}$$

and, according to the prospective method,

14. It has been assumed in the last paragraph that t embraces an integral (i.e. exact) number of $\frac{1}{m}$ths of a year. If, however, t is an integral number of *years*, it may be shown that $_tV^{(m)}_{\overline{n}|} = {}_tV_{\overline{n}|}$.

From formula (48),

$V^{(m)} \quad P^{(m)} \quad (m)$

$$= v^{n-t} - P_{\overline{n}|} \times a_{\overline{n-t}|} \text{ since } P^{(m)}_{\overline{n}|} \times a^{(m)}_{\overline{1}|} = P_{\overline{n}|}$$
$$= {}_tV_{\overline{n}|}$$

That is to say, the policy value at the end of an integral number of years of a sinking fund assurance effected at premiums payable more frequently than yearly is the same

as if the assurance had been effected at annual premiums; this is so because the instalment premiums received in each complete year are the exact equivalent of an annual premium.

Surrender Values and Paid-up Policies

15. If a sinking fund assurance were to be discontinued by the policy-holder, it might be thought that he should be allowed a surrender value of the full amount of the policy value. For various practical reasons, however, surrender values cannot be allowed on such a favourable basis, some deduction being necessary from the full theoretical policy value. It is, therefore, customary to calculate the surrender values of sinking fund assurances effected by periodic premiums on some such bases as the following, viz.—

(*a*) 92½% of the premiums paid after the first year, accumulated at 3% per annum compound interest; or

(*b*) The whole of the premiums paid after the first year, accumulated at 3% per annum compound interest, together with the accumulated amount of the excess (if any) of the first year's premium over a stated proportion of the sum assured, usually about £2%.

A common basis of surrender value for single premium assurances is 92½% of the single premium paid, accumulated at compound interest at 3% per annum.

It is usual for the appropriate surrender value basis to be stated in the policy itself, and it will have been noted that custom favours a retrospective mode of calculation, as being more likely to be understood by the policy-holder than a prospective method.

Instead of surrendering outright, a policy-holder who wished to discontinue payment of premiums could have the policy converted into a paid-up policy. Such a policy would be free from payment of further premiums and would carry a correspondingly reduced sum assured. The sum assured under the paid-up policy would be that which would be secured by applying the surrender value of the policy as a single premium under a new assurance for the balance of the original term. For example, if S be employed to denote the surrender value of a policy effected t years ago to secure a certain sum assured at the expiration of n years from the original date of assurance, the corresponding paid-up policy would be $\frac{S}{v^{n-t}}$ or $S(1 + i)^{n-t}$.

16. The paid-up policy corresponding to a policy value of

${}_tV_{\overline{n}|}$ is denoted by ${}_tW(A_{\overline{n}|})$, and from the preceding paragraph it will be evident that

$$ {}_tW(A_{\overline{n}|}) = {}_tV_{\overline{n}|} \times (1 + i)^{n-t} \quad . \quad . \quad . \quad (49)$$

Many other forms may be devised for the theoretical paid-up policy, but it will be sufficient for present purposes merely to consider the two following—

(*a*) If a sinking fund assurance of 1 payable at the end of n years were effected subject to the payment of annual premiums during the first t years only, the annual premium would clearly be $\dfrac{v^n}{\mathrm{a}_{\overline{t}|}}$. The payment of t yearly premiums of $\dfrac{v^n}{\mathrm{a}_{\overline{t}|}}$ would thus entitle the policy-holder to a fully paid assurance for a sum assured of 1, payable at the end of n years from the original date of assurance.

A policy-holder who has paid t years' premiums of $P_{\overline{n}|}$ under an ordinary sinking fund assurance should, therefore, be entitled by way of paid-up policy to a sum assured of

$$ {}_{\overline{n}|} \qquad {}_{\overline{t}|} - \mathrm{a}_{\overline{t}|} \qquad\qquad P_{\overline{n}|} = \frac{v^n}{}$$

$$\frac{a_{\overline{t}|}}{a_{\overline{n}|}} \text{ since } \mathrm{a}_{\overline{t}|} =$$

$$\text{and } \mathrm{a}_{\overline{n}|} =$$

So that ${}_tW(A_{\overline{n}|}) = \dfrac{{}_{\overline{t}|}}{a_{\overline{n}|}}$ . . . . . . (50)

(*b*) The payment of an annual premium of $P_{\overline{n}|}$ for n years will secure a sum assured of unity at the end of the term. Assume that after t years' premiums have been paid, the policy-holder decides (1) to convert the policy into a paid-up assurance, and (2) to effect a new policy maturing at the end of a further $(n - t)$ years, subject to payment of exactly the same annual premium as before, viz. $P_{\overline{n}|}$.

It can easily be shown that under the new policy the sum assured will be $\dfrac{P_{\overline{n}|}}{P_{\overline{n-t}|}}$. Now, the total of the sums assured under the paid-up and new policies must be the same as under the original policy, because no alteration has been made in the premiums payable by the policy-holder, nor in the rate

Therefore, since the sum assured under the original policy is 1, and the sum assured under the new policy is $\frac{P_{\overline{n}|}}{P_{\overline{n-t}|}}$, the sum assured under the paid-up policy must be the balance, viz. $1 - \frac{P_{\overline{n}|}}{P_{\overline{n-t}|}}$.

$$\text{So that } {}_tW(A_{\overline{n}|}) = 1 - \frac{P_{\overline{n}|}}{P_{\overline{n-t}|}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

As a matter of interest it may be added that all the forms for ${}_tW(A_{\overline{n}|})$ may be shown to be algebraically identical.

When premiums are payable more frequently than yearly, the formula corresponding to formula (49) is

$${}_tW(A^{(m)}_{\overline{n}|}) = {}_tV^{(m)}_{\overline{n}|} \times (1 + i)^{n-t} \quad . \quad . \quad . \quad . \qquad (52)$$

from which paid-up policy values may readily be calculated in practice, especially if the policy value ${}_tV^{(m)}_{\overline{n}|}$ has already been obtained.

In this paragraph it has been assumed that the full policy value would be applied, without any deduction, to secure the paid-up policy value. In practice, however, the general rule would be to apply the surrender value only, as stated in paragraph 15.

17. It may be instructive to give an example of the calculation of a paid-up policy according to the three formulas obtained in the last paragraph. For this purpose, suppose that it is required to find the theoretical paid-up policy which could be granted at the end of 8 years under an assurance effected at annual premiums to secure a sum assured of £100 payable at the expiration of 20 years, as in the case of the example in paragraph 12.

From formula (49), the paid-up policy

$$= £33{\cdot}09 \times (1{\cdot}03)^{12} = £33{\cdot}09 \times 1{\cdot}42576 = £47{\cdot}18.$$

From formula (50) it is £100 $\frac{a_{\overline{8}|}}{a_{\overline{20}|}}$ at 3%

From formula (51) it is £100 $(1 - \frac{P_{\overline{20}|}}{P})$

EXERCISE VI

IN answering these Questions, the tables at the end of the book may be used.

1. Calculate, on the basis of an effective rate of interest of 3% per annum, the single premium for a sinking fund assurance of £100 to mature at the end of 25 years.

2. Find the annual, half-yearly, and quarterly premiums to secure a sinking fund assurance of £250 to mature at the end of 12 years; the basis of the calculations is to be an effective rate of interest of 3½% per annum.

3. Find the policy values of the above sinking fund assurances at the end of 5 years, using the respective rates of interest underlying the original calculation of the premiums.

4. An insurance company allows surrender values of sinking fund assurances effected by annual premiums on the basis of 92½% of the premiums due and paid after the first year, accumulated with 3% per annum compound interest, together with the accumulated excess of the first year's premium over £2% of the sum assured, the accumulation being based on a rate of interest of 2% per annum. Calculate the surrender value of a 15-year sinking fund assurance for £350, effected 5 years ago at annual premiums of £18 5s. 5d., all of which have been paid to date.

5. A 30-year sinking fund policy was effected 6 years ago for a sum assured of £500, subject to half-yearly premiums of £5 2s. 10d., which were calculated at an effective rate of interest of 3% per annum. Find the amount of the paid-up policy which could, in theory, be allowed if no further premiums are paid. All the premiums which have fallen due have been paid.

6. Ten years ago a sinking fund policy was effected for a sum assured of £100 payable at the expiration of 25 years, subject to annual premiums of £2 13s. 4d. The policy-holder now desires to extend the original term by 5 years without altering the sum assured. Calculate, on the basis of an effective rate of interest of 3% per annum, what future annual premium should be charged for the next 20 years.

7. What would have been the future annual premium in the case of the last question, if the policy-holder desired to pay only 15 years' further premiums?

CHAPTER VII

YIELDS

1. In the preceding chapters it has been shown how the financial terms of different types of transaction may be fixed on the basis of a given rate of interest. In practice, however, it frequently happens that the terms of a transaction are known, and it is desired to ascertain what rate of interest is involved. For example, an investor contemplating the purchase of a stock redeemable 25 years hence at £105 and bearing a rate of interest of 4% per annum may be informed that the market price is £97, and he would probably require to know what rate of interest would be secured to him by purchasing at that price. Again, a person who wished to raise a sum of money on mortgage of house property might well require to ascertain the rates of interest underlying the various terms of repayment by instalments which are available to him. In each of these representative cases the terms of the transaction are fixed, and the problem is to ascertain the corresponding rate of interest.

2. Generally speaking, it is not possible to solve these problems by direct methods of calculation. An exception exists, however, when the transaction takes the form of the exchange of a present amount for a larger sum due at a future date, as, for example, in the case of a sinking fund assurance effected at a single premium or of the purchase of a savings certificate.

In these exceptional cases the problem of determining the rate of interest may always be reduced to finding the value of i from an equation of the form $(1 + i)^n = C$, where n and C are known. As a practical illustration, consider the case in which a 30-year sinking fund assurance would be granted by an insurance company at a rate of single premium of £41 4s. 0d. per cent, and a prospective policy-holder desired to know what rate of interest he would earn in respect of such a policy. Let the required rate of interest per unit per annum be denoted by i.

Then $41{\cdot}2 \times (1 + i)^{30} = 100$

And $\qquad (1 + i)^{30} = 100 \div 41{\cdot}2.$

This may be described as the fundamental equation.

Taking logarithms of both sides of this equation,

$$30 \times \log_{10}(1+i) = \log_{10} 100 - \log_{10} 41{\cdot}2$$
$$= 2 - 1{\cdot}6148972 = {\cdot}3851028$$

Therefore $\log_{10}(1+i) = {\cdot}0128368$

Whence, on taking antilogarithms,

$$(1+i) = 1{\cdot}03$$

And $i = {\cdot}03$

3. The reason why direct calculation cannot always be employed to find the rate of interest involved in a given transaction is that the appropriate fundamental equation might not be capable of solution by any known algebraic method. Even if it were merely required to ascertain what rate of interest were involved when an annuity-certain of 1 per annum had a given present value, the impossibility of determining i from an equation of the type $a_{\overline{n}|} = C$ would prevent a solution of the problem by direct calculation.

In view of the serious limitations on the method of direct calculation in this respect, the general method of solution which is adopted in practice is to use compound interest tables inversely. So far, these tables have been employed to find the values of various functions corresponding to given rates of interest; by using the tables inversely the rate of interest may be found from the value of the function itself and the term. Incidentally, it may be mentioned that, if necessary, the tables could also be used to find the term, having given the rate of interest and the value of the function.

In practice, therefore, the above sinking fund problem would be solved from the fundamental equation

$$(1+i)^{30} = 100 \div 41{\cdot}2, \text{ or } 2{\cdot}4272$$

by searching the compound interest tables at successive rates of interest so as to find what rate of interest produces a value of $2{\cdot}4272$ for $(1+i)^n$ when $n = 30$. By this means, the desired rate of interest would be found to be 3% per annum.

The method of using compound interest tables inversely in this way is of such importance in practical work that a further example will be given of its application. Suppose that a prospective mortgagor has been offered a loan of £1,000 to be repayable in 30 quarterly instalments of £41 12s. 9d., including principal and interest, and that he desires to know what rate of interest he is being called upon to pay.

At a nominal rate of interest j convertible 4 times a year the quarterly instalment would be $\frac{£1,000}{a_{\overline{30}|}}$, calculated at rate $\frac{j}{4}$.

And $\frac{1,000}{a_{\overline{30}|}} = 41\cdot638$

Whence the fundamental equation is

$$a_{\overline{30}|} = \frac{1,000}{41\cdot638} = 24\cdot017$$

On searching the tables it is found that $a_{\overline{30}|} = 24\cdot017$ (almost exactly) when the rate of interest is $1\frac{1}{2}\%$. Since this rate corresponds to $\frac{j}{4}$, the terms of the proposed mortgage are based on a nominal rate of interest of 6% per annum, convertible quarterly. To find the equivalent effective rate, the value of $(1\cdot015)^4$ would be ascertained from the tables, and unity deducted therefrom, thus giving the result of $6\cdot136\%$ or £6 2s. 9d.%, approximately.

The device of working on the assumption of a nominal rate of interest should be carefully noted. Where any of the payments in the original transaction are made with greater frequency than yearly, it is usually found simplest to make the calculations on the basis of a nominal rate of interest convertible with the same frequency. If necessary, the corresponding effective rate may be finally obtained from the nominal rate.

To revert to the general question of the inverse use of compound interest tables, it does not often happen that the exact required value of the function is found in the tables, because the desired rate of interest might lie between two adjacent tabulated rates. Even in such cases, however, the method provides the basis of a solution, as will be seen later.

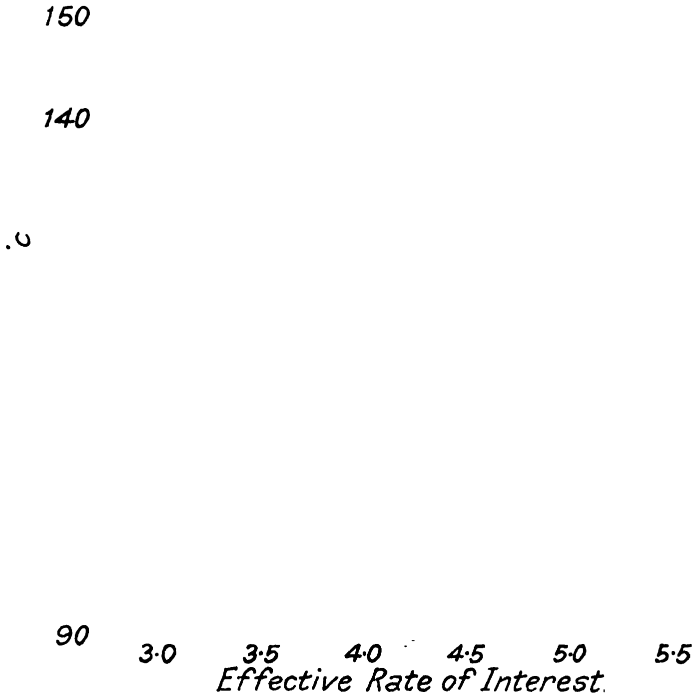
Graphic Process

4. An obvious alternative mode of ascertaining the rate of interest underlying any given transaction is by means of the graphic process, which will now be illustrated in finding the yield from a redeemable security, purchased at a given price.

Suppose that it is desired to find the yield on a debenture, redeemable at 105 forty years hence, and bearing yearly interest of 5% per annum, the first interest payment being due at the end of one year, and the purchase price being 113.

The first step is to value the security at several rates of interest in turn, using for that purpose the formula

$$105 \times v^{40} + 5 \times a_{\overline{40|}}.$$



The results of these valuations are as follows–

Effective Rate of interest per annum.	Value of Debenture. (Purchase Price)
3 %	147·8
$3\frac{1}{2}$%	133·3
4 %	120·8
	110·1
5 %	100·7
	92·6

By plotting these results graphically in the usual way, the above " curve " is obtained.

By means of this graph, the yield corresponding to any given purchase price between 92·6 and 147·8 may be immediately obtained by inspection ; at a price of 113 the debenture is seen to yield about 4·36% per annum.

The graphic method of solution is not, however, adopted in practice because of the labour which is involved in constructing the graph, and the difficulty which is often experienced in reading off the desired yield with sufficient accuracy.

5. These objections may be overcome to a very large extent if it be assumed that the "curve" is a straight line between two points which respectively correspond to tabulated rates of interest on either side of the desired yield. In these circumstances there would be only two preliminary values to be calculated, one of which would be larger than the given value, and the other smaller. If the graph were then plotted on a large scale, the method would be reasonably serviceable for practical purposes. In the interests of accuracy, however, the preliminary values should be calculated at rates of interest as close together as the available compound interest tables permit. If possible, these rates of interest should differ by no more than $\frac{1}{8}$ per cent.

The diagram on page 78, which is based on the above graph, shows how a solution of the original problem is simplified by the assumption that the curve is a straight line between points B and D.

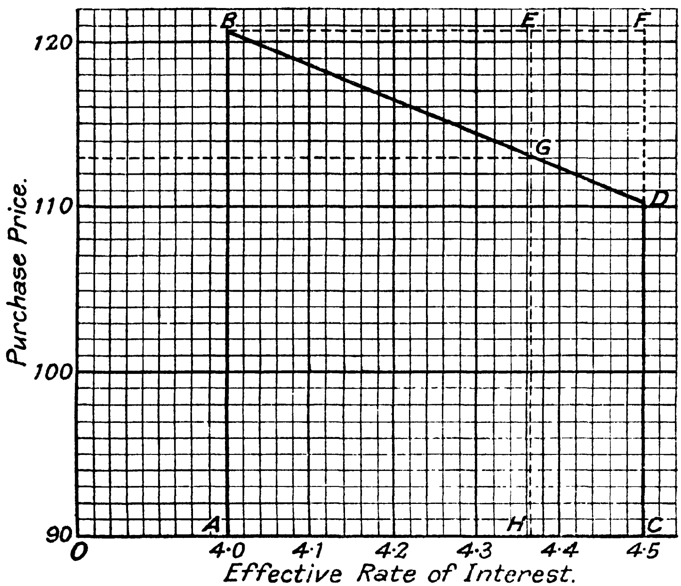
In this diagram AB represents the value of the debenture (120·8) at a rate of interest of 4% per annum, represented by OA ; CD represents the value of the debenture (110·1) at a rate of interest of $4\frac{1}{2}$% per annum, represented by OC, and GH represents the given purchase price of 113. OH, therefore, represents the required yield, which is seen to be about 4·36% per annum, as before.

Method of Interpolation

6. It will now be shown, however, that on the assumption that the curve is a straight line between points B and D, there is really no need to solve the problem graphically at all. First, let BF be drawn parallel to OC meeting CD produced at F, and let HG be produced to meet BF at E. Then the desired yield $OH = OA + BE$.

Since the triangles BEG and BFD are similar, the ratio of BE to EG is the same as the ratio of BF to FD. But BE represents the increase in yield which arises from a

reduction of EG in the purchase price; and BF represents the increase in yield which arises from a reduction of FD in the purchase price. Hence, between the prescribed limits of purchase price, the increase in the yield is always proportional to the decrease in the price. This provides the key to the arithmetical solution of the problem. From the table in paragraph 4, it is seen that a reduction in the price from 120·8 to 110·1 is accompanied by an increase in the yield



from 4% to $4\frac{1}{2}$%; it is now desired to know what increase in the yield would arise as a result of reducing the price from 120·8 to 113·0.

The arithmetical working may be set out as follows—

A price reduction of 10·7 increases the yield by ·5%.

Therefore a price reduction of 1 increases the yield by $\frac{\cdot 5}{10\cdot 7}\%$

And a price reduction of 7·8 increases the yield by $\frac{\cdot 5}{10\cdot 7} \times 7\cdot 8\%$

$= \cdot 36\%$

Whence the desired yield = 4·36% as before.

The general process of finding intermediate values of tabulated quantities is known as interpolation. The arithmetical method here illustrated involves only two tabulated values, and is known as first difference interpolation. If the values of the tabulated quantities are not widely different, first difference interpolation usually gives results which are sufficiently accurate for practical purposes.

7. In the case examined in paragraph 5 above, the tabulated values *decrease* when the rate of interest is *increased*. In some cases, however, the tabulated values would *increase* with an *increase* in the rate of interest, but it may easily be shown on the lines of paragraph 6 that the method of first difference interpolation could still be applied. The general principle underlying the application of first difference interpolation to these problems is, therefore, that the increase (or decrease) in the tabulated value is proportional to the corresponding increase in the rate of interest.

For example, suppose that the rate of annual premium for a 25-year sinking fund assurance is £2 12s. 0d. per cent of the sum assured, and that it is desired to find the rate of interest which would be yielded by such a policy. Here $2{\cdot}6\ (s_{\overline{26}|} - 1) = 100$.

Whence the fundamental equation is $s_{\overline{26}|} = 39{\cdot}4615$.

On reference to the tables, $s_{\overline{26}|}$ is seen to be 38·5530 when the rate of interest is 3% per annum, and 41·3131 when the rate of interest is $3\frac{1}{2}$% per annum.

Thus, an *increase* of 2·7601 in $s_{\overline{26}|}$ corresponds to an *increase* in the rate per cent of ·5

And an *increase* of ·9085 in $s_{\overline{26}|}$ corresponds to an *increase* in the rate per cent of $\dfrac{\cdot 9085}{2{\cdot}7601} \times \cdot 5 = \cdot 165$

So that the desired rate of interest = 3·165% per annum.

If compound interest tables were available giving values of "*s*" for rates of interest differing by only $\frac{1}{8}$%, a much more accurate result would be obtainable.

8. It has thus been shown that the necessary steps to ascertain the rate of interest underlying a given transaction are—

(*a*) To determine the fundamental equation: this must be of such a form that one side consists of compound interest functions, and the other consists of a numerical value.

(*b*) To evaluate the functions in the fundamental equation

at two rates of interest as close together as the available compound interest tables permit; these rates must be so selected that one result of the valuation is in excess of the given numerical value, and the other is in defect of that value.

(c) To ascertain by first difference interpolation between the results derived under (b), what rate of interest corresponds to the given numerical value in the fundamental equation.

9. It is not always easy to decide upon the rates of interest at which the preliminary valuations should be made in order to comply with the requirements stated under (b) in the last paragraph. At worst, however, it can be merely a matter of successive trials. In any event, after the first valuation has been made, it is easy to see whether a higher or lower rate of interest must be employed to produce a result on the other side of the given numerical value. In this connexion it must be remembered that an increase in the rate of interest decreases present values, and increases amounts, irrespective of whether such present values or amounts refer to single payments or to series of payments.

If, however, the original problem relates to a redeemable security, an approximation to the yield can be obtained by means of one of the formulas given in the next paragraph. Such an approximate result would serve to indicate suitable rates of interest for the preliminary valuations.

Approximate Formulas Applicable to Redeemable Securities

10. Let R = the redemption value of the security,

D = the dividend payable in each interval, the first being due one interval hence,

n = the number of intervals at the end of which the security is redeemable,

V = the given purchase price of the security,

and i = the effective rate of interest per unit per interval yielded by the security at a price of V.

The problem is to find i from R, D, n, and V.

Consider, in the first instance, the case where V is greater than R, i.e. the case where the security is purchased at a premium.

(a) The capital at the date of purchase is V, and this decreases gradually (see Chapter IV, paragraph 8) until by the end of the n intervals it becomes R. The *average* decrease in capital value per interval is thus $\frac{V-R}{n}$.

But the total decrease in capital value, viz. $V - R$, must be met out of the dividends received, and it is only the balance of the dividends that may properly be regarded as interest. The *average* interest per interval is therefore

Now the capital, it is assumed, decreases by an equal amount each interval from V at the beginning of the first interval to $R + \frac{V-R}{n}$ at the beginning of the last interval, so that the *average* amount of capital invested is midway between these two amounts. This average capital may be shown to be

$$R + \frac{n+1}{2n}(V - R)$$

The required effective yield per unit of capital per interval is given by dividing the average interest per interval by the average capital invested.

$V - R$

Thus

$2n$

If both numerator and denominator in this expression be divided by R, the formula becomes

k

$2n$

since $\frac{D}{R} = g$, and $\frac{V-R}{R} = k$ (See Chapter IV, paragraph 6).

The formula is usually stated in the second form, although the first is more convenient for its practical application. It will be noted that the formula does not involve compound interest functions, but it cannot be relied upon to furnish very accurate results, especially if n should be large.

(*b*) If the capital invested is regarded as V throughout the entire period, the amount to be set aside as a sinking fund payment each interval to provide for the capital "loss" on redemption of $V - R$ would be $\frac{V-R}{[illegible]}$.

The balance of the dividend, viz. $D - \frac{V - R}{s_{\overline{n}|}}$, should alone be regarded as interest, and since the capital is V,

$$i = \frac{D - \frac{V - R}{s_{\overline{n}|}}}{V}$$

On dividing both numerator and denominator by R, as before,

$$i = \frac{\quad}{1 + k}$$

One point calls for mention here, however, in that $s_{\overline{n}|}$ should, strictly speaking, be calculated at rate i per interval. If it were so calculated, the result of the formula would be exact, but if a trial rate were used (as would be the case in practice) the formula would give approximate results only. The result of one application of the formula may, however, be employed as a trial rate to calculate $s_{\overline{n}|}$ for a second application, and if this were done, the result of the second application would usually be very near the truth.

11. For the purpose of evolving the above formulas, it has been assumed that the security stands at a premium on its redemption price. The formulas apply, however, without modification, if the security stands at a discount, but in their application it must be carefully observed that the numerical values of both $(V - R)$ and k would be negative in such circumstances.

That the first formula applies when the security is purchased at a discount will be seen if it is borne in mind (1) that $\frac{R - V}{n}$ would be added to the periodic dividend to give the average total interest applicable to each interval, and (2) that the average amount of capital invested would be the mean between V at the beginning of the first interval and $R - \frac{R - V}{n}$ at the beginning of the last interval.

That the second formula applies also if the security stands at a discount is clear from the consideration that the premium on redemption of $R - V$ may be considered as the equivalent of interest of $\frac{R - V}{s_{\overline{n}|}}$ per interval. Hence the total interest

applicable to each interval is $D + \dfrac{R - V}{s_{\overline{n}|}}$, which is, of course, equal to $D - \dfrac{V - R}{s_{\overline{n}|}}$ included in the formula.

12. The principles underlying these two approximate formulas will now be applied to find the yields from debentures standing respectively at a discount and at a premium on their redemption value.

In each case it will be supposed that the debenture is redeemable at 105 in 20 years time, and that it bears a rate of interest of 5% per annum payable half-yearly; so that $R = 105$, $n = 40$, and $D = 2{\cdot}5$.

(*a*) If the purchase price were 120, the debenture thus standing at a premium of 15 on its redemption value, the following would be the respective calculations—

(i) *First Formula.*

$$\begin{aligned}
&\text{Average half-yearly interest} && = 2{\cdot}5 - \frac{15}{40} \\
& && = 2{\cdot}125 \\
&\text{Capital invested at beginning of first half-year} && = 120 \\
&\text{Capital invested at beginning of last half-year} && = 105 + \frac{15}{40} \\
& && = 105{\cdot}375 \\
&\text{Whence, the mean capital invested} && = \tfrac{1}{2}(120 + 105{\cdot}375) \\
& && = 112{\cdot}6875
\end{aligned}$$

$$\therefore \quad = \frac{2{\cdot}125}{112{\cdot}6875} =$$

So that at a price of 120 the debenture yields approximately 3·772% per annum convertible half-yearly.

(ii) *Second Formula.*

Discount on redemption $= 15$

Corresponding half-yearly sinking fund instalment $= \dfrac{15}{s_{\overline{40}|}}$

If $s_{\overline{40}|}$ is calculated at a trial rate of 2%, $\dfrac{15}{s_{\overline{40}|}} = \dfrac{15}{60{\cdot}4020}$

$$= {\cdot}248 \text{ approx.}$$

Deducting this from the half-yearly dividend of 2·5, the result is 2·252, which, divided by the price of 120, gives ·01877. So that at a price of 120 the debenture yields approximately 3·754% per annum convertible half-yearly.

If $s_{\overline{40}|}$ had been calculated at a trial rate of $1\frac{3}{4}\%$ the resulting yield would have been $3\cdot730\%$ per annum convertible half-yearly. The true yield in this case is $3\cdot742\%$ per annum convertible half-yearly, and the superiority of the second formula over the first from the point of view of accuracy is thus apparent.

(*b*) If the purchase price were 98, the debenture thus standing at a discount of 7 on its redemption value, the following would be the respective calculations—

(i) *First Formula.*

$$\text{Average half-yearly interest} = 2\cdot5 + \frac{7}{40}$$
$$= 2\cdot675$$
$$\text{Capital invested at beginning of first half-year} = 98$$
$$\text{Capital invested at beginning of last half-year} = 105 - \frac{7}{40}$$
$$= 104\cdot825$$
$$\text{Whence mean capital invested} = \tfrac{1}{2}(98 + 104\cdot825)$$
$$= 101\cdot4125$$
$$\text{Therefore} \quad i = \frac{2\cdot675}{101\cdot4125} = \cdot02638.$$

So that at a price of 98 the debenture yields approximately $5\cdot276\%$ per annum convertible half-yearly.

(ii) *Second Formula.*

$$\text{Premium on redemption} = 7$$
$$\text{Corresponding half-yearly sinking fund instalment} = \frac{7}{s_{\overline{40}|}}$$
$$\text{If } s_{\overline{40}|} \text{ is calculated at a trial rate of } 2\tfrac{1}{2}\%, \frac{7}{s_{\overline{40}|}} = \frac{7}{67\cdot4026}$$
$$= \cdot104 \text{ approx.}$$

Adding this to the half-yearly dividend of 2·5, the result is 2·604, which, divided by the price of 98, gives ·02657. So that at a price of 98 the debenture yields approximately $5\cdot314\%$ per annum, convertible half-yearly.

If $s_{\overline{40}|}$ had been calculated at a trial rate of $2\frac{3}{4}\%$, the resulting yield would have been $5\cdot302\%$ per annum convertible half-yearly. The true yield is $5\cdot307\%$ per annum convertible half-yearly, and here again, as is usually the case, the second formula produces the more accurate result.

13. If a redeemable security is purchased at a premium on its redemption value, and a sinking fund has to be set up at a specified rate of interest (not necessarily that yielded by the

security), as mentioned in Chapter IV, paragraph 10, the reasoning of that paragraph may be used to find the yield in such circumstances.

On dividing both sides of formula (37) by V,

$$i = \frac{D -}{} V$$

where $s'_{\overline{n}|}$ is calculated at the rate of interest per unit per interval at which the sinking fund payments are to be actually invested. It will be seen that this formula is the same in type as that obtained in paragraph 10 (*b*) of the present chapter, and need not, therefore, be further considered here.

14. In the case of the redeemable securities dealt with in this chapter, it has been assumed that the security is purchased immediately after a dividend due-date. If, as is usually the case, the security is purchased on some other date, slight modifications will become necessary in the methods of calculating the yield. If the yield is to be calculated by the method of interpolation summarized in paragraph 8, it is merely necessary to make the preliminary valuations of the security as at the actual date of purchase by accumulating the value on the last previous due-date, as mentioned in Chapter IV, paragraph 11. If, however, an approximation to the yield is to be obtained by means of the formulas of paragraph 10 of the present chapter, the first step must clearly be to obtain a price as on a dividend due-date. In practice, this is sometimes done by deducting the accrued dividend from the quoted purchase price, such accrued dividend being a proportion of the dividend corresponding to the period which has elapsed since the last dividend due-date. This modified price is then regarded as the equivalent purchase price immediately after payment of the next succeeding dividend. For instance, if the debenture in paragraph 12 had been purchased when the unexpired period was 19 years 7 months, five months' dividend would have already accrued. The amount of this accrued dividend would be $\frac{5}{12} \times 5 = 2{\cdot}083$, and this amount would be deducted from the present purchase price to give the price as on the next succeeding dividend due-date. The value of n to be used in the calculation of the yield would then be 39 instead of 40, but the method of calculation itself would remain unchanged.

The effect of estimating the price on the next dividend due-date by this method is that the amount of dividend which will accrue from the date of purchase up to the next dividend due-date is regarded as interest on the actual purchase price. If the security stood at a premium on its redemption value, a part of this accruing dividend would, in fact, be utilized to write down the capital value of the security on the next succeeding dividend due-date. On the other hand, if the security stood at a discount on its redemption value, the capital value of the security would be correspondingly written up. Thus, the estimated price of the security on the next dividend due-date is too great if the security stands at a premium, and too small if the security stands at a discount. The tendency is, therefore, for the yield to be respectively under-stated, and over-stated.

To obtain the price, as at the next dividend due-date, which *exactly* corresponds to the true yield, it would be necessary to add to the present price interest calculated from the date of purchase up to the next dividend due-date, and to deduct from the result the full amount of the next dividend. The rate of interest employed should be that actually yielded by the security, but an estimate would be sufficient for practical purposes. For example, if the debenture in paragraph 12 were purchased at 120 when it had an unexpired term of 19 years 8 months, the calculation of the equivalent price as on the next dividend-due date would be as follows—

Given purchase price	120
Add 2 months' interest up to next dividend due-date, at trial rate of 4% per annum	·8
	120·8
Deduct next dividend	2·5
Equivalent price immediately after payment of next dividend (when unexpired term is 19 years 6 months) .	118·3, approx.

If required, the calculation of the yield could then be made by any of the methods already discussed in this chapter.

EXERCISE VII

In answering these Questions the tables at the end of the book may be used.

1. A debenture of nominal value £100, bearing interest at the rate of 5% per annum, payable half-yearly, is redeemable at £106 at the end of 15 years. Find the yield to a purchaser at a price of £98,

assuming that the first payment of interest is due at the end of 6 months.

2. A sinking fund assurance of £1,500 maturing at the end of 25 years from the date of assurance has been effected at an annual premium of £39 5s. 0d. Find the rate of interest which is yielded to the policy-holder.

3. A loan of £2,500 is being repaid by 30 equal yearly instalments of £158 2s. 3d., including principal and interest. What rate of interest is the mortgagor paying ?

4. A debenture of nominal value £100, bearing interest at the rate of 4% per annum, payable half-yearly, and redeemable at £102 at the end of 3 years, is purchased for £97. Find, without using compound interest tables, the approximate effective yield from the investment.

5. A 6% corporation stock of nominal value £100, which is redeemable in 25 years, and under which dividends are payable half-yearly, is purchased at £108. The purchaser requires to set up a sinking fund to replace the premium on the redemption value, but can do so only at a rate of interest of 4% per annum, convertible half-yearly. What rate of interest would the purchaser derive from the investment in these circumstances ?

6. A loan of £1,250 is being repaid by 20 equal half-yearly instalments of £81 17s. 6d., including principal and interest. What rate of interest is yielded to the mortgagee if he decides to set up a sinking fund at a rate of interest of 4% per annum, convertible half-yearly ?

7. Find the effective yield from the purchase of the following debenture at a price of £94 on the 15th May, 1924—

Nominal value . . .	£100
Redemption value . .	£103
Date of redemption . .	31st December, 1934
Rate of interest payable . .	4% per annum, payable half-yearly on 30th June and 31st December.

CHAPTER VIII

THE EFFECT OF INCOME TAX

1. In view of the very considerable bearing which taxation has upon all financial matters, the present chapter will be devoted to considering the effect of income tax upon the various types of transaction which have already been dealt with. In this connexion, although income tax will be specifically referred to, it must be understood that the following remarks would also apply as regards super-tax. It is, of course, no purpose of this textbook to expound in detail the circumstances in which liability for income tax arises and how such liability is assessed; a discussion of the incidence of the tax in a few representative cases is probably all that is required for immediate purposes.

As a general rule, income tax is chargeable on interest, dividends, rents, profits, gains, annuities, etc., of all descriptions. If, therefore, a sum of money were deposited at a fixed rate of interest, the depositor would be liable for income tax on each instalment of interest received. Also, if a person purchased an annuity-certain, he would be liable for income tax on the whole of each instalment of annuity, irrespective of the fact that each of such instalments consists partly of a return of capital. An exception occurs, however, where the annuity arises under a mortgage transaction where the debt is repayable in instalments which include principal and interest; in such a case income tax is chargeable only on so much of each instalment as represents interest. Another important exception to the general rule stated above is that profits arising from capital appreciation are not taxable. If, therefore, a redeemable security were purchased at a discount on the redemption value, the only liability for income tax would be in respect of the periodic dividends; no income tax would be chargeable at maturity in respect of the difference between the purchase price and the redemption value, since this would be regarded as capital appreciation. If the security were purchased at a premium on the redemption value, income tax would, however, still be payable on the full amount of the dividends, despite the fact that these consist, in some part, of return of capital.

2. A rate of interest which is subject to deduction on

account of income tax is usually described as a gross rate, and one which is not subject to income tax (either because of exemption or because the liability for tax has already been satisfied) is known as a net rate.

Where capital is invested at a fixed gross rate of interest payable in each year, the net rate actually receivable by the investor may be calculated by multiplying the gross rate by $(1-t)$, where t represents the rate of income tax per unit of interest; but this simple method of calculation applies only where the capital repayable is identical with the sum invested, or, alternatively, where the interest is payable in perpetuity, as, for example, in the case of an irredeemable security. Subject to these conditions, the gross rate corresponding to a given net rate could be found by dividing the latter by $(1-t)$. If, for instance, a sum were placed on deposit at a gross rate of interest of 4% per annum, and income tax were at the rate of 4s. 6d. in the £ (i.e. at the rate of ·225 per unit), the corresponding net rate of interest would be $4(1-\cdot 225)$ or 3·1% per annum. Also, if a loan were being arranged on the basis of a net rate of interest of 5% per annum, and the rate of income tax were 4s. 6d. in the £ as before, the transaction would be the same in effect as if the loan were granted at a gross rate of interest of $(5 \div \cdot 775)$ or 6·452% per annum.

It is most important to point out, however, that if there should be a difference between the principal invested and the sum ultimately repayable (as in the case of a redeemable security bought at a price differing from its redemption value) these simple relationships do not exist between the relative net and gross interest yields. This is because the liability for income tax would be based, not on the gross yield itself, but on the interest (or dividend) income actually receivable year by year.

Valuation of Annuities-Certain Allowing for Tax

3. The valuation of annuities-certain with allowance for income tax will now be considered. For this purpose an immediate annuity-certain of 1 per annum payable yearly for n years will be taken, and it will be assumed that income tax will be at the rate of t per unit throughout the term of the annuity.

Since income tax would be chargeable upon the whole of each annuity payment, a purchaser would receive a net

annual payment of $(1-t)$ from the annuity. The present value of the annuity at an effective rate of interest of, say, i per unit per annum would therefore be $(1-t) \times a_{\overline{n}|}$ calculated at rate i. Now it is important to observe that the valuation rate of interest i is a net rate, since it represents the rate which the purchaser would earn after all liability for income tax had been satisfied. In the great majority of instances, valuations are actually required at a net rate of interest, which is, in fact, the real rate of interest which is operative; from the point of view of the investor, the fact that interest is apparently at a higher (gross) rate is of little significance, apart from the possibility of changes in future rates of income tax. If there is a liability for income tax, but for special reasons an annuity-certain is to be valued at a *gross* rate of interest, the calculation should be made at the equivalent net rate of interest.

In illustration of these remarks, an immediate annuity-certain of £50 per annum payable yearly for 25 years will be valued, first at a net rate of interest of $4\frac{1}{2}\%$ per annum, and afterwards at a gross rate of interest of 5% per annum, assuming in each case that the rate of income tax chargeable is 4s. 6d. in the £. The calculations are as follows—

Gross annuity per annum . . .	£50	0	0
Less income tax at 4s. 6d. in the £ .	11	5	0
Net annuity per annum . . .	£38	15	0

Present value of annuity at a net rate of interest of $4\frac{1}{2}\%$ per annum $= £38{\cdot}75 \times a_{\overline{25}|}$ at $4\frac{1}{2}\% = £38{\cdot}75 \times 14{\cdot}8282$

$= £574{\cdot}593 = £574$ 11s. 10d.

Present value of annuity at a gross rate of interest of 5% per annum $= £38{\cdot}75 \times a_{\overline{25}|}$ at a *net* rate of interest of $(1 - {\cdot}225) \times 5\%$ per annum, i.e. at $3\frac{7}{8}\%$ per annum

$= £38{\cdot}75 \times 15{\cdot}8306 = £613{\cdot}436 = £613$ 8s. 9d.

In making valuations of immediate annuities-certain with allowance for income tax, the rule is, therefore, to value only the net amount of each annuity instalment, and to employ a net valuation rate of interest. This rule applies in all cases, irrespective of the frequency either of the payment of the annuity instalments or of the convertibility of the valuation rate of interest; it also clearly applies as regards deferred annuities.

4. It has already been mentioned that where a loan is being liquidated by means of an annuity-certain, income tax is chargeable on so much only of each instalment as represents

interest. In such a case, therefore, the net rate of interest receivable by the lender could be directly calculated by multiplying the gross rate by $(1-t)$, where t represents the rate of income tax per unit. Changes in the rate of income tax would, of course, have the effect of varying the net rate of interest receivable by the lender from time to time. Advances of this kind are always made at a gross rate of interest, and the impossibility of forecasting rates of income tax would prevent the arrangement of such a transaction on the basis of a uniform *net* rate of interest.

Redeemable Securities

5. To allow for the effect of income tax in the valuation of a redeemable security, the valuation would be made by one of the methods discussed in Chapter IV, but the net dividends only would be valued instead of the gross dividends. If the rate of income tax were t per unit throughout the period of the investment, the appropriate allowance for income tax could easily be made by respectively substituting $(1-t) \times D$ for D, and $(1-t) \times g$ for g where these occur in the formulas of that chapter. The valuation rate of interest employed would be a net rate, as the liability for income tax would have already been fully allowed for in the formula.

The following practical illustration will suffice to make the position clear. Suppose that it is desired to find the value as on 30th August, 1924, of a 5% debenture redeemable at 102 on 31st December, 1948, half-yearly dividends being due on 30th June and 31st December in each year; the value to be calculated so as to yield a net rate of interest of 4% per annum, convertible half-yearly, on the assumption that income tax will be chargeable throughout at the rate of 4s. 6d. in the £.

The value (to yield 4% per annum, convertible half-yearly) as on the previous interest due-date (viz. 30th June, 1924) would be

$$102 \times v^{49} + (1 - \cdot 225) \times 2{\cdot}5 \times a_{\overline{49}|} \text{ calculated at } 2\%$$
$$= 102 \times v^{49} + 1{\cdot}9375 \times a_{\overline{49}|}$$
$$= 98{\cdot}817.$$

And the value after a further 2 months would be

$$(1 + \frac{2}{12} \times \cdot 04) \times 98{\cdot}817$$
$$= 1{\cdot}00667 \times 98{\cdot}817$$
$$= 99{\cdot}476 = £99\ 9s.\ 6d. \text{ approx.}$$

6. As regards the calculation of yields from redeemable securities, the effect of income tax would be allowed for on the same general principles as those which have already been mentioned. If the method of interpolation were employed, the preliminary valuations would be made, allowing for the incidence of income tax on the lines of the last paragraph ; the resulting yield would, of course, be net. If either of the methods underlying the formulas of Chapter VII, paragraph 10, were employed, it would be necessary to substitute the net dividend of $(1-t) \times D$ for the gross dividend of D, or the net rate of dividend $(1-t) \times g$ for the gross rate of dividend g, as the case might be, whilst the $s_{\overline{n}|}$ of the second formula would be calculated at a net rate of interest. It will be evident that in each instance the resulting yield would be net.

7. A practical point may here be mentioned. It has been assumed in this chapter that the rate of income tax will remain unchanged throughout the duration of each transaction. It is hardly probable that in practice the course of events would justify such an assumption. If the future rates of income tax were exactly known, the net amounts of the future annuity instalments, dividends, or other payments could be calculated, and their total present value could be ascertained by dealing separately with each. A forecast of the rates of income tax might, it is true, be attempted, but the usual practice is to assume that the current rate of income tax will continue throughout the duration of each transaction. It would be well, however, in individual instances, to ascertain the effect of assuming different uniform rates of income tax, as by this means general conclusions may often be formed as to what the relative effects of future income tax changes are likely to be upon different investments.

EXERCISE VIII

In answering these Questions the tables at the end of the book may be used.

N.B.—The rate of income tax is to be taken as 4s. in the £ throughout each of the following transactions.

1. Find the present value at an effective net rate of interest of 4% per annum of an immediate annuity-certain of £28 6s. 8d. (gross) per annum, payable half-yearly for 26 years.

2. Find the present value at a gross rate of interest of 5% per annum, payable half-yearly, of a deferred annuity-certain of £15 3s. 8d. (gross) per annum, payable quarterly for 14 years, the first payment being due $5\frac{1}{4}$ years hence.

3. A loan of £550 is to be repaid in 20 half-yearly instalments, including principal and interest, and the lender requires to obtain a net rate of interest of 4% per annum payable half-yearly. Find the amount of each half-yearly instalment.

4. A debenture of nominal value £100, bearing interest (gross) at the rate of 4½% per annum, payable half-yearly, and redeemable at 103 at the end of 5 years is purchased for £98. (*a*) Find, without using compound interest tables, the approximate effective net yield from the investment. (*b*) Check the result by using compound interest tables.

5. A redeemable security of nominal value £100, which is redeemable in 20 years at 104, and under which the gross dividends are at the rate of 6% per annum, payable half-yearly, is purchased for £110. Find the nominal net yield, payable half-yearly, which would be derived by the purchaser, assuming that he sets up a sinking fund at a gross rate of interest of 5% per annum, convertible half-yearly, to replace the premium on the redemption value.

6. A sinking fund policy for £250, maturing at the end of 10 years, has just been effected by annual premiums of £20 18s. 9d. At what gross rate of interest would the policy-holder have to invest these annual sums of £20 18s. 9d. and the interest accumulations in order to obtain the same result as under the policy ? (*Note.*—There is no liability for income tax in respect of the sum assured payable at the maturity of the sinking fund policy.)

7. Find the effective net yield derivable from the purchase of the following debenture at a price of £97 on 28th October, 1924—

Nominal value . . .	£100
Redemption value . .	£104
Date of redemption . .	31st January, 1930
Gross rate of interest payable .	5½% per annum, payable half-yearly on 31st January and 31st July

Check the result by means of an approximate formula.

CHAPTER IX

CONSTRUCTION AND PROPERTIES OF COMPOUND INTEREST TABLES

1. THE values of $(1 + i)^n$, v^n, $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are usually tabulated in compound interest tables for all integral values of n up to 50, or sometimes 100. Occasionally, however, a value of one of these functions is required for a value of n which is outside the range of the tables; this usually happens when the rate of interest is a nominal one which is convertible at frequent intervals during the year. For instance, the amount of a unit of capital might be required at the end of 30 years, calculated at a nominal rate of interest of 4% per annum, convertible quarterly, and the value of $(1{\cdot}01)^{120}$ would in all probability be outside the range of the tables.

In cases such as this, the desired value could always be obtained from the appropriate formula with the aid of logarithms, but if the value of the function is required at a rate of interest given in the tables, it is not necessary to resort to this means, as expedients are available whereby the tabulated values may themselves be utilized to provide results.

2. The amount of a unit of capital for a term of $(x + y)$ years is $(1 + i)^{x+y}$, and by the Laws of Indices, this is equal to $(1 + i)^x \times (1 + i)^y$. Correspondingly, the present value of a unit of capital due at the end of $(x + y)$ years is

$$v^{x+y} = v^x \times v^y.$$

These relationships provide a simple means of finding the amount or present value of a unit of capital where the term is beyond the range of the tables. For example, suppose that it is desired to find the value of $(1{\cdot}025)^{80}$ from the tables at the end of this book; the required value would be obtained by subdividing the 80 intervals into two periods, each within the scope of the tables, and by multiplying together the corresponding two values of $(1{\cdot}025)^n$. Thus,

$$(1{\cdot}025)^{80} = (1{\cdot}025)^{50} \times (1{\cdot}025)^{30} \text{ or } (1{\cdot}025)^{40} \times (1{\cdot}025)^{40}, \text{ etc.}$$

Similarly, the present value of a unit of capital due at the end of 60 intervals would be given by $v^{50} \times v^{10}$, $v^{40} \times v^{20}$, etc.

DIAGRAMMATIC EXPLANATION

3. Although this portion of the subject may be dealt with exclusively from the mathematical point of view, it is usually

found helpful to explain it diagrammatically in the following manner.

Suppose that the value of a compound interest function is desired in respect of a total period of $(x + y)$ intervals, where both x and y are individually within the range of the tables. Let the line AC represent the total period $(x + y)$, AB representing x, and BC representing y.

Point A thus represents the present time, point B represents the moment when the period of x expires, and point C represents the moment when the further period of y expires.

(a) *To find the amount of a unit of capital at the end of* $(x + y)$ *intervals.*

A unit of capital invested at the present time (i.e. at point A) will have accumulated to $(1 + i)^x$ by the end of the period x (i.e. at point B). But each unit of capital then standing invested will accumulate to $(1 + i)^y$ by the time point C is reached. So that since there will be $(1 + i)^x$ units of capital at point B, there will finally be $(1 + i)^x \times (1 + i)^y$ units by the end of the total period of $(x + y)$.

Hence $\qquad (1 + i)^{x+y} = (1 + i)^x \times (1 + i)^y$

(b) *To find the present value of a unit of capital due at the end of* $(x + y)$ *intervals.*

Here the unit of capital is due at point C, and the "present" value at point B is therefore v^y. Each unit of capital due at point B has a present value at point A of v^x, and, since there are v^y units of capital due at point B, their actual present value at point A is $v^y \times v^x$.

Hence $\qquad v^{x+y} = v^x \times v^y$

(c) *To find the present value of an immediate annuity-certain of* 1 *per interval payable for* $(x + y)$ *intervals.*

This annuity may be subdivided into two annuities, one of which would be payable during the period AB, the other being payable during the period BC. If, therefore, the present values of these two annuities are ascertained as at point A, and added together, the result will represent the desired present value of the original annuity.

Now the present value at point A of the first annuity is clearly $a_{\overline{x}|}$ and the " present value " at point B of the second annuity is $a_{\overline{y}|}$.

It is now necessary to ascertain the present value of the second annuity at point A instead of at point B. It has just been shown that this annuity is equivalent to a capital sum of $a_{\overline{y}|}$ due at point B. Since each unit of capital due at point B has a present value at point A of v^x, the present value of the $a_{\overline{y}|}$ units of capital due at point B is therefore $v^x \times a_{\overline{y}|}$, which thus represents the present value at point A of the second annuity. The present value of the original annuity is, therefore, $a_{\overline{x}|} + v^x \times a_{\overline{y}|}$.

Hence $$a_{\overline{x+y}|} = a_{\overline{x}|} + v^x \times a_{\overline{y}|}.$$

This formula is clearly correct, since $a_{\overline{x+y}|} = a_{\overline{x}|} + {}_{x|}a_{\overline{y}|}$.

(d) *To find the amount of an immediate annuity-certain of 1 per interval payable for* $(x + y)$ *intervals.*

As in case (c) above, the annuity may be subdivided into two annuities, one being payable during the period AB, the other being payable during the period BC. If, therefore, the amounts of each of these two annuities are ascertained as at point C, and the results are added together, the desired amount of the original annuity will be obtained.

Now the amount of the second annuity at point C is clearly $s_{\overline{y}|}$ and the amount *as at point* B of the first annuity is similarly $s_{\overline{x}|}$.

It is now necessary to ascertain the amount of the first annuity as at point C instead of as at point B. It has just been shown that this annuity is equivalent to a capital sum of $s_{\overline{x}|}$ due at point B. Since each unit of capital due at point B will accumulate to $(1 + i)^y$ by the time point C is reached, it follows that $s_{\overline{x}|}$ units of capital due at point B will accumulate to $(1 + i)^y \times s_{\overline{x}|}$ by the end of the original period of $(x + y)$ intervals. This amount of $(1 + i)^y \times s_{\overline{x}|}$ thus represents the amount of the first annuity as at point C. The amount of the original annuity as at point C is therefore

Hence $$s_{\overline{x+y}|} = s_{\overline{y}|} + (1 + i)^y \times s_{\overline{x}|}.$$

4. If any of the functions on the right-hand side of the above equations should also happen to be outside the range

of the tables, as would necessarily be the case if the total period $(x + y)$ exceeded twice the longest term embraced by the tables, the same principles would have to be employed to evaluate the function in question.

For example, if the value of $s_{\overline{210}|}$ were required, but the maximum period covered by the tables were 100, the working would be as follows—

$$s_{\overline{210}|} = s_{\overline{100}|} + (1 + i)^{100} \times s_{\overline{110}|}$$
$$s_{\overline{110}|} = s_{\overline{100}|} + (1 + i)^{100} \times s_{\overline{10}|}$$

Whence, substituting this value of $s_{\overline{110}|}$ in the former equation,

$$s_{\overline{210}|} = s_{\overline{100}|} + (1 + i)^{100} \times s_{\overline{100}|}$$

Results identical with those given by this method would, of course, be secured by subdividing the original period into three, or more, periods (each within the range of the tables) and proceeding on the lines of paragraph 3.

5. If the desired value of any function should not be obtainable from compound interest tables, because the rate of interest itself is not embraced in the tables, there are three courses which may be taken. The desired value may be calculated from the fundamental formula, or it may be calculated approximately by interpolating between the tabulated values of the function at adjacent rates of interest; or, finally, if extensive use is likely to be made of that particular rate of interest, complete tables may be specially constructed.

It may be useful to give an example of the method of approximate calculation by interpolation. For this purpose, suppose that the value of v^{30} is required at $3\frac{7}{8}\%$ per annum, and that the only tables available are those given at the end of this book.

The calculation would be as follows—

$$\begin{aligned} v^{30} \text{ at } 3\tfrac{1}{2}\% &= \cdot 35628 \\ v^{30} \text{ at } 4\% &= \underline{\cdot 30832} \\ \text{Difference corresponding to } \tfrac{1}{2}\% &= \cdot 04796 \\ \text{Whence difference corresponding to } \tfrac{3}{8}\% &= \tfrac{3}{4} \times \cdot 04796 \\ &= \cdot 03597 \\ \text{And } v^{30} \text{ at } 3\tfrac{7}{8}\% &= \cdot 35628 - \cdot 03597 \\ &= \cdot 32031 \end{aligned}$$

This value is, of course, approximate only, the true value being ·31965.

As a further example, suppose that the value of $s_{\overline{50}|}$ is required at an effective rate of interest of $3\frac{1}{8}\%$ per annum.

If the nearest rates of interest given in the tables were 3% and $3\frac{1}{2}\%$, the calculation would be as follows—

$$
\begin{aligned}
s_{\overline{50}|} \text{ at } 3\% &= 112{\cdot}7969 \\
s_{\overline{50}|} \text{ at } 3\tfrac{1}{2}\% &= \underline{130{\cdot}9979} \\
\text{Difference corresponding to } \tfrac{1}{2}\% &= 18{\cdot}2010 \\
\text{And difference corresponding to } \tfrac{1}{8}\% &= \tfrac{1}{4} \times 18{\cdot}2010 \\
&= 4{\cdot}5502 \\
\text{Whence} \qquad s_{\overline{50}|} \text{ at } 3\tfrac{1}{8}\% &= 112{\cdot}7969 + 4{\cdot}5502 \\
&= 117{\cdot}3471
\end{aligned}
$$

The true value of $s_{\overline{50}|}$ at $3\frac{1}{8}\%$ is 117·0555, and this example serves to show that, unless the difference between the adjacent values of the function is small, the approximate result obtained by the method is likely to be subject to an appreciable error. The difference between the adjacent values employed should therefore be kept as small as possible by using tables in which the successive rates of interest differ by a small margin.

The Continued Method

6 If it should be necessary to construct tables of a compound interest function at any given rate of interest, what is known as the "continued method" would be employed, whereby the value of each function tabulated is calculated directly from the preceding value. Thus, in calculating a table of $(1 + i)^n$, the value of $(1 + i)^2$ would be calculated from the value of $(1 + i)$, the value of $(1 + i)^3$ would be calculated from the value of $(1 + i)^2$, and so on until the table was complete. This method obviously entails the use of a "working formula," by means of which each value is in turn employed to give the next. Before the "working formula" can be applied, however, there must be an "initial value" upon which to operate, for the first value cannot be ascertained otherwise than by direct calculation. Finally, there must be some convenient means of checking results, especially since an error in any particular value would render all subsequent values inaccurate. This risk of perpetuating errors is sometimes made a point of criticism, but

steps may be taken to minimize the defect by calculating in advance every tenth value of the function; an automatic check then arises in the course of the calculations as each tenth value is reached. It should not be overlooked, however, that it is an advantage of the method that if any value is proved correct (by independent calculation), it follows that all the preceding values are also correct.

7. The significance of the foregoing remarks will be the more readily appreciated if a short illustration is given of the practical working of the method. For this purpose, the construction will be undertaken of a table of $(1 + i)^n$ at a rate of interest of 5% for values of n from 1 to 20. It will be supposed that the results are required correct to four places of decimals, so that to ensure the desired degree of accuracy, it will be necessary to make the calculations in the first instance to six places of decimals. In this case, the working formula would be $(1 + i)^{n+1} = (1 + i)^n \times (1 + i)$, which shows that, once the value of the function has been calculated for any particular duration, the value of the function corresponding to the next higher duration may be ascertained by multiplying by $(1 + i)$. For instance, if the value of $(1 + i)^{11}$ at 5% was 1·710339, the value of $(1 + i)^{12}$ could be immediately obtained therefrom by multiplying by 1·05. In this way the value of $(1 + i)^{12}$ would be found to be 1·795856.

The initial value in this instance is clearly $(1 + i)$, or 1·05. Applying the working formula to this initial value, the value of $(1 + i)^2$ is given as $(1{\cdot}05) \times (1{\cdot}05) = 1{\cdot}102500$. The value of $(1 + i)^3$ is then obtained from $1{\cdot}102500 \times 1{\cdot}05 = 1{\cdot}157625$, from which, in turn, the value of $(1 + i)^4$ is found by multiplying by 1·05 and so on, until the table is completed. In practice, the working would be set out in schedule form, as shown below, and as a safeguard against error, the values of $(1{\cdot}05)^{10}$ and $(1{\cdot}05)^{20}$ would be independently calculated by logarithms, and inserted in the schedule before the continued method was commenced. The successive multiplications by 1·05 would be performed very conveniently on a calculating machine if one were available, in which case the schedule would consist of two columns only, headed respectively n and $(1 + i)^n$. In the absence of a calculating machine, the working formula might possibly be modified to $(1 + i)^{n+1} = (1 + i)^n + i \times (1 + i)^n$, which shows that, instead of multiplying the appropriate value of $(1 + i)^n$ by $(1 + i)$ to obtain $(1 + i)^{n+1}$, the same result is to be obtained by

adding a year's interest to $(1 + i)^n$ The schedule is as follows—

n	$(1 + i)^n$	$i \times (1 + i)^n$
Col. (1)	Col. (2)	Col. (3)
1	1·050000	·052500
2	1·102500	·055125
3	1·157625	·057881
4	1·215506	·060775
5	1·276281	·063814
6	1·340095	·067005
7	1·407100	·070355
8	1·477455	·073873
9	1·551328	·077566
10	1·628894	·081445
11	1·710339	·085517
12	1·795856	·089793
13	1·885649	·094282
14	1·979931	·098997
15	2·078928	·103946
16	2·182874	·109144
17	2·292018	·114601
18	2·406619	·120331
19	2·526950	·126348
20	2·653298	

The schedule would be commenced by inserting in Col. (2) the initial value of 1·05 as well as the two values of $(1{\cdot}05)^{10}$ and $(1{\cdot}05)^{20}$ which had already been calculated. The first value in Col. (3) would then be calculated by multiplying the first value in Col. (2) by ·05, thus giving ·052500. The second value in Col. (2) would then be obtained by adding this figure of ·052500 to the previous value in Col. (2), viz. 1·050000, thus giving 1·102500, from which the next value in Col. (3) would be obtained by multiplying by ·05, when the cycle of operations would recommence. As soon as the value of $(1{\cdot}05)^{10}$ was reached, its value as thus ascertained would be compared with that already inserted in the schedule; if the two values agreed, the work would be continued, but, otherwise, the error would be traced to its source before proceeding to the calculation of $(1{\cdot}05)^{11}$ and the subsequent values.

If, when the schedule was complete, the value of $(1{\cdot}05)^{20}$

agreed with that already calculated, the whole of the previous values should be correct. It is, however, advisable to place a further check upon the work, especially if a calculating machine has been used, as there would then be an appreciable risk that figures had been incorrectly transcribed. The check referred to consists of finding the total of the figures in column (2) and comparing the result with that obtained by employing an algebraic method of summation.

In the case of a column of $(1+i)^n$, the total of any number of values could be obtained by the use of the formula for summing a geometric progression. In the present instance, the series is $(1{\cdot}05)$, $(1{\cdot}05)^2$, etc., up to and including $(1{\cdot}05)^{20}$, the sum of which is $(1{\cdot}05) \times \dfrac{(1{\cdot}05)^{20}-1}{(1{\cdot}05)-1}$. It may be pointed out that in the present case it is hardly necessary to regard the matter from the point of view of a geometric progression, since the sum of all the given values of $(1{\cdot}05)^n$ is clearly equivalent to the amount of an annuity-certain of 1 per annum, payable yearly in advance for 20 years, calculated at an effective rate of interest of 5% per annum.

The total of column (2) of the schedule, as obtained by actual addition, is 34·719246, and the value of $(1{\cdot}05) \times \dfrac{(1{\cdot}05)^{20}-1}{\cdot 05}$, calculated with the aid of seven figure logarithms, is 34·71926. The slight discrepancy is principally due to the last decimal place in the latter result being unreliable, owing to the use of logarithms correct only to seven significant figures, whilst a further small error is probably introduced owing to the sixth place of decimals in each of the values summed being approximately correct only. The check shows, however, that the calculations are correct to 4 places of decimals, and that is all that is required in the circumstances.

8. If it were desired to construct a table of v^n, the continued method could be utilized, employing an initial value of v in conjunction with a working formula $v^{n+1} = v^n \times v$.

This working formula is, however, not very convenient to apply in practice, since the continued multiplication by v is rather awkward, especially in the absence of a calculating machine. Accordingly, the device is adopted of starting at the other end of the table, and working backwards through the various powers of v by multiplying successively by $(1+i)$. The initial value would then correspond to the highest value of n which is to be tabulated, and the working formula would be $v^{n-1} = v^n \times (1+i)$. The initial value of v^n would of

course be calculated by logarithms. As a safeguard against perpetuating errors, it would be well also to calculate by logarithms every tenth value of the function v^n before commencing the construction of the table by the continued method. If the calculations had to be made without a machine, it would probably be easiest to adapt the working formula on the lines referred to in the previous paragraph. In such circumstances, the working formula would be

$$v^{n-1} = v^n + i \times v^n.$$

Since the total of all the values from v to v^t is $a_{\overline{t}|}$, or $\frac{1 - v^t}{i}$, the accuracy of the final figures could be checked by comparing the result of this formula with the total of the figures themselves. The calculations should, of course, be performed throughout to two more decimal places than are required in the final results.

9. Once tables of v^n and $(1 + i)^n$ have been constructed, it is a simple matter to form tables of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ by addition, because $a_{\overline{n}|} = v + v^2 + v^3 + \ldots + v^{n-1} + v^n$,

and $s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + \ldots + (1+i)^{n-2} + (1+i)^{n-1}$

In passing, it should be noted that $s_{\overline{n}|}$ does not consist of the sum of the first n values of the function $(1 + i)^n$, but of the sum of the first $(n - 1)$ values of that function, increased by unity.

The labour of constructing tables of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ in this way may be minimized by employing the working formulas

and $s_{\overline{n+1}|} = s_{\overline{n}|} + (1 + i)^n$.

For purposes of illustration, a few values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ will be found at 3%, assuming that the necessary values of v^n and $(1 + i)^n$ have been given.

For the table of $a_{\overline{n}|}$ the working would be—

n	v^n	$a_{\overline{n}|}$
Col. (1)	Col. (2)	Col. (3)
1	·97087	·97087
2	·94260	1·91347
3	·91514	2·82861
4	·88849	3·71710
5	·86261	4·57971

The first value of $a_{\overline{n}|}$ is simply v; the second value is found by adding v^2 to the first value, the third value being found by adding v^3 to the second value, and so on, in accordance with the above working formula.

In the case of a table of $s_{\overline{n}|}$ the working would be—

n	$(1 + i)^n$	$s_{\overline{n}\|}$
Col. (1)	Col. (2)	Col. (3)
1	1·03000	1·00000
2	1·06090	2·03000
3	1·09273	3·09090
4	1·12551	4·18363
5	- -	5·30914

The first value of $s_{\overline{n}|}$ is unity; the second value is found by adding $(1 + i)$ to the first value, the third being found by adding $(1 + i)^2$ to the second value, and so on, in accordance with the above working formula.

The values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ as obtained above would not be reliable beyond four places of decimals.

The check formula for the sum of the first t values of $a_{\overline{n}|}$ is $\frac{1}{i}(t - a_{\overline{t}|})$ in the application of which the value of $a_{\overline{t}|}$ would be independently calculated. This check formula is derived as follows—

The sum of the first t values of $a_{\overline{n}|}$

$a_{\overline{2}|}$

$\frac{-v}{i}$ $+ \frac{1 - v^{t-1}}{i} + \frac{1 - v^t}{i}$

$= \frac{1}{i} \} t \times 1 - (v$

The check formula for the sum of the first t values of $s_{\overline{n}|}$ is $\frac{1}{i}$ in the application of which the value

of $s_{\overline{t+1}|}$ would be independently calculated. This check formula is derived as follows—

The sum of the first t values of $s_{\overline{n}|}$

$$\frac{(1+i)-1}{i} + \frac{(1+i)^2-1}{i} + \frac{(1+i)^3-1}{i} + \dots + \frac{(1+i)^t-1}{i}$$

10. If it is desired to construct a table of $s_{\overline{n}|}$ without first compiling a table of $(1+i)^n$, the work may be conveniently performed by the continued method, as the following simple relationship exists between $s_{\overline{n}|}$ and $s_{\overline{n+1}|}$, viz. $s_{\overline{n+1}|} = (1+i) \,.\, s_{\overline{n}|} + 1$. This relationship may be proved algebraically, but from general reasoning it is clear that the amount of an annuity-certain of 1 per annum, payable yearly for $(n+1)$ years, is equivalent to the amount of an annuity-certain of 1 per annum, payable yearly for n years, accumulated for a further year, and increased by the final payment of unity due at the end of $(n+1)$ years.

To construct the tables, the following would therefore be used—

Initial value: $s_{\overline{1}|} = 1$

Working formula: $s_{\overline{n+1}|} = (1+i) \,.\, s_{\overline{n}|} + 1$

Check formula (for sum of first t values): $\frac{1}{i}\left\{s_{\overline{t+1}|} - (t+1)\right\}$,

as in the previous paragraph.

In practice, the work would be set out in schedule form, as in the case of the following illustration, where the first five

values of $s_{\overline{n}|}$ are calculated at an effective rate of interest of 4% per annum.

n	$s_{\overline{n}|}$	$i \,.\, s_{\overline{n}|}$
Col. (1)	Col. (2)	Col. (3)
1	1·00000	·04000
2	2·04000	·08160
3	3·12160	·12486
4	4·24646	·16986
5	5·41632	—
TOTAL	15·82438	

The method of working is to insert unity in column (2) as the first value of $s_{\overline{n}|}$; then

$$
\begin{aligned}
&= 1{\cdot}00000 + {\cdot}04000 + 1 \\
&= 2{\cdot}04000
\end{aligned}
$$

and
$$
\begin{aligned}
s_{\overline{3}|} &= s_{\overline{2}|} + i \,.\, s_{\overline{2}|} + 1 \\
&= 2{\cdot}04000 + {\cdot}08160 + 1 \\
&= 3{\cdot}12160,
\end{aligned}
$$

and so on.

The check formula for these five values would be

$$|-6) = \frac{1}{{\cdot}04}\left\{\frac{(1{\cdot}04)^6 - 1}{{\cdot}04}\right.$$

the result of which would be found with the aid of seven-figure logarithms to be 15·8244, thus proving the correctness of the tabulated values of $s_{\overline{n}|}$.

In practice, every tenth value of $s_{\overline{n}|}$ would be independently calculated by means of logarithms and inserted in the schedule before the continued method was begun. Also, the work would be performed to two more decimal places than it was desired to retain in the final results.

11. If it is desired to construct a table of $a_{\overline{n}|}$ without first compiling a table of v^n, the continued method would be used, and, as in the case of v^n, the initial value would be that corresponding to the highest value of n to be tabulated.

In this case, the initial value (to be calculated by logarithms) would be

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i};$$

the working formula would be $a_{\overline{n-1|}} = (1+i) \times a_{\overline{n|}} - 1$; and the check formula for finding the sum of the first t values of $a_{\overline{n|}}$ from $a_{\overline{1|}}$ to $a_{\overline{t|}}$ inclusive would be $\frac{1}{i}(t - a_{\overline{t|}})$, as in paragraph 9.

The working formula may be interpreted as showing that, if the present value of an n-year annuity-certain of 1 per annum be accumulated for 1 year and the payment then due be deducted, the result will be the value as at the end of the year of the remaining $(n-1)$ payments.

As an illustration of the actual method of construction, a few values of $a_{\overline{n|}}$ will be calculated at a rate of interest of $3\frac{1}{2}\%$ per annum, commencing with the value where $n = 50$.

By logarithms the initial value $a_{\overline{50|}}$ is 23·4556.

The following schedule shows the working—

n	$a_{\overline{n|}}$	$i \,.\, a_{\overline{n|}}$
Col. (1)	Col (2)	Col. (3)
50	23·4556	·8209
49	23·2765	·8147
48	23·0912	·8082
47	22·8994	·8015
46	22·7009	—

It would not be possible to apply the check formula given above until the whole of column (2) had been completed. It may be added, however, that, if necessary, suitable check formulas applicable to a section of any of the tables of $(1+i)^n$, v^n, $s_{\overline{n|}}$, or $a_{\overline{n|}}$ could be easily devised on the same lines as the check formulas which have already been considered.

For example, in the present instance, the sum of the 5 tabulated values would be represented by—

$$+ \frac{1 - v^{47}}{i} + \quad - +$$

$$= \frac{1}{i} \{ 5 - \qquad + v^{50}) \}$$

$$\text{or } \frac{1}{i}(5 - 1 \quad \times a_{\overline{5|}}$$

Logarithmic Construction

12. If a calculating machine is not available, and the rate of interest is such that continued multiplication by $(1 + i)$ would be unduly laborious, tables of $(1 + i)^n$ and v^n would be constructed with the aid of logarithms.

In the case of a table of $(1 + i)^n$, the initial value would be $\log (1 + i)$, and since $\log (1 + i)^2 = 2 \log (1 + i)$, $\log (1+i)^3 = 3 \log (1 + i)$, etc. the working formula would be

$$\log (1 + i)^{n+1} = \log (1 + i)^n + \log (1 + i).$$

In applying the method, every tenth value of the function $\log (1 + i)^n$ should be calculated in advance, and inserted in the schedule. The accuracy of the logarithmic portion of the work could be checked by agreeing the sum of the first t values of $\log (1 + i)^n$ with the result of the algebraic formula $\frac{t(t+1)}{2} \times \log (1 + i)$.

Once the column of $\log (1 + i)^n$ had been constructed and checked, it would be necessary to find the corresponding antilogarithms. It is here that the method is somewhat vulnerable, as errors are very apt to occur, and it is essential to check each conversion from logarithms to antilogarithms separately. This could be done either by having this portion of the work done independently by another operator, or by re-converting the antilogarithms into logarithms and subsequently comparing the resulting logarithms with those originally found by the continued method. After the final antilogarithms had been checked, the usual summation check would be applied to the resulting values of $(1 + i)^n$.

The construction of a table of v^n by means of logarithms would proceed on similar lines, except that the initial value would correspond to the highest value of n to be tabulated; the working formula would be

$$\log v^{n-1} = \log v^n + \log (1 + i)$$

and the column of $\log v^n$ would thus be completed by continued addition of $\log (1 + i)$. The total of the first t values of $\log v^n$ from $\log v$ to $\log v^t$ inclusive could be checked from the algebraic formula

$$\times \log v, \text{ or } -\frac{t(t+1)}{2} \times \log (1 + i).$$

The antilogarithms of the column $\log v^n$ would be checked

on the same lines as those referred to above, and the final summation check on the values of v^n would be as described in paragraph 8.

In constructing tables of $(1+i)^n$ and v^n by means of logarithms, seven-figure logarithm tables cannot be relied upon to provide results correct to more than five significant figures; this is so because any slight error in the last decimal place of $\log(1+i)$ is successively increased as the value of n is increased. For this reason, the logarithms of $(1+i)$ have been calculated up to as many as 15 places of decimals for all the more usual values of i which arise in practice. A table of the values of $\log(1+i)$ is given in the Introduction to Chambers' Tables, calculated to 10 places of decimals for rates of interest varying from $\frac{1}{4}\%$ to 6% per annum by quarter per cent. intervals.

EXERCISE IX

1. Use the tables at the end of the book to find the values of v^{70}, $a_{\overline{65}|}$, $(1+i)^{83}$ and $s_{\overline{57}|}$ at an effective rate of interest of $3\frac{1}{2}\%$ per annum.

2. By means of the tables at the end of the book, find, approximately, the value at an effective rate of interest of $4 \cdot 7\%$ per annum of (a) v^{38} and (b) $s_{\overline{14}|}$.

3. (a) Construct, correct to 4 places of decimals, tables of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ corresponding to an effective rate of interest of $3\frac{1}{4}\%$ per annum, for values of n from 1 to 5 inclusive.

(b) Devise formulas for checking the results by summation.

4. Construct, correct to 4 places of decimals, tables of v^n and $(1+i)^n$ corresponding to an effective rate of interest of $2 \cdot 4\%$ per annum for values of n from 1 to 5 inclusive.

5. Obtain formulas for finding the sum of 15 values of v^n, $(1+i)^n$, $a_{\overline{n}|}$ and $s_{\overline{n}|}$ respectively, corresponding to values of n from 26 to 40 inclusive.

MISCELLANEOUS EXERCISES

(Taken from Chartered Insurance Institute Examination Papers)

IN answering the following Questions compound interest tables must *not* be used.

1. Given tables of $s_{\overline{n}|}$ at 2%, 3%, and 4% for all values of n from 1 to 40, show how to find the amount at the end of 30 years, at a nominal rate of interest of 4%, convertible half-yearly, of a life assurance premium paid half-yearly for 30 years.

2. A loan of £100 at 5% interest is to be repaid in 4 years by equal annual payments containing principal and interest. Calculate the annual payment and construct a schedule of (*a*) capital contained in each payment, and (*b*) interest contained in each payment. Given v^4 at 5% = ·8227. (The calculation should be limited to three places of decimals only.)

3. Given that, at $3\frac{1}{2}$% interest, the value of an annuity of 1 per annum, payable at the beginning of each year for 12 years, is 10·00, find the sum which, paid annually at the end of each year will accumulate to 1 in 12 years.

4. Given $a_{\overline{20}|}$ at 6% interest = 11·47, find the price which must be given for a debenture of £100, due at par at the end of 20 years, and bearing interest at the rate of 4% payable yearly, in order that the purchaser may realize 6% on his investment. Assume that the first payment of interest is due at the end of one year.

5. A bond for £100, redeemable at par at the end of two years, and bearing interest at the rate of 4% per annum, payable half-yearly, is bought for £95. What is the gross rate of interest realized by the investor ? Prove your answer.

6. A loan of £855 10s. 0d. is to be discharged in 20 years by an annuity which gives the lender 5% per annum on the amount of the loan for the whole 20 years, and enables him to replace the amount of the loan by a sinking fund accumulated at 4% per annum. The amount of 1 at 4% per annum for 20 years = 2·19112. What is the amount of the annuity required to fulfil the above conditions ?

7. What would be the purchase price of a perpetual ground rent of £152 per annum if the investment is made in order to provide interest at 5% per annum ?

8. What yearly payment at the end of each one of the next 5 years is equivalent to £202 11s. 4d. payable now ? Interest to be reckoned at 4% per annum.

9. Find the present value at a rate of interest of 3% per annum of a yearly annuity-certain of £30 per annum payable for 20 years, and deferred 10 years. Given v^{10} at 3% = ·74409.

10. It is proposed to grant a loan of £250 at 6% per annum, to be repaid by three equal annual payments, including principal and interest. Calculate the amount of the annual payment, and construct a schedule showing the division of each payment into principal and interest. (Results to nearest penny.) $a_{\overline{3}|}$ at 6% = 2·67301.

11. Assuming compound interest tables to be available, how would you ascertain the price which an investor could give, as on 1st April, 1924, for the following security in order to obtain a yield of $4\frac{1}{2}\%$ per annum convertible half-yearly ?—

£100 National War Bond, bearing interest at the rate of 5% per annum, payable half-yearly on 1st April and 1st October, and redeemable on 1st April, 1928, at £105.

Ascertain from first principles the effective rate of interest which would be yielded by the security. (Ignore income tax throughout.)

12. A sinking fund policy securing £1,000 at the end of 20 years has been granted at an annual premium of £35 12s. 7d. Ascertain approximately the rate of interest involved, given $s_{\overline{21}|}$ at $3\% = 28{\cdot}676$ and $s_{\overline{21}|}$ at $3\frac{1}{4}\% = 29{\cdot}460$.

APPENDIX

(*a*) To show that if a redeemable security is purchased at a premium or at a discount on its redemption value, the periodic sinking fund instalments will be sufficient to replace the difference between the purchase price and the redemption value.

Employing the general case described in paragraph 4 of Chapter IV, the present value of the security at an effective rate of interest of i per unit per annum

$$= R\,(v^n + g \times a_{\overline{n}|}^{(p)})$$

$$= R\left\{v^n + \frac{g}{j_{(p)}}(1 - v^n)\right\}$$

Now, the interest for $\frac{1}{p}$th of a year on a unit of capital

$$= (1 + i)^{\frac{1}{p}} - 1 = \frac{j_{(p)}}{p}.$$

Therefore, the interest for $\frac{1}{p}$th of a year on the original purchase price of the security

$$= \frac{j_{(p)}}{p} \cdot R\left\{v^n + \frac{g}{j_{(p)}}(1 - v^n)\right\}$$

$$= \frac{j_{(p)}}{p} \cdot Rv^n + \frac{Rg}{p} - \frac{Rgv^n}{p}$$

$$= \frac{Rg}{p} - \frac{Rv^n}{p}\,(g - j_{(p)})$$

And the dividend for $\frac{1}{p}$th of a year $= \frac{Rg}{p}$, so that the balance of each dividend, after deducting interest for $\frac{1}{p}$th of a year on the purchase price

$$= \frac{Rv^n}{p}\,(g - j_{(p)})$$

If this last amount were set aside each interval as a sinking fund, the accumulations by the end of n years would amount to

$$Rv^n\,(g - j_{(p)})\,s_{\overline{n}|}^{(p)}$$

$$= R\,(g - j_{(p)})\,a_{\overline{n}|}^{(p)} \text{ since } v^n\,s_{\overline{n}|}^{(p)} = a_{\overline{n}|}^{(p)}$$

$$= R\left\{g \times a_{\overline{n}|}^{(p)} - j_{(p)} \times a_{\overline{n}|}^{(p)}\right\}$$

$$= R\left\{g \times a_{\overline{n}|}^{(p)} - (1 - v^n)\right\} \text{ since } a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{j_{(p)}}$$

$$= R\,(v^n + g\,a_{\overline{n}|}^{(p)}) - R$$

which represents the difference between the redemption value and the original purchase price.

Further, the amount of the sinking fund accumulations by the end of m years (m being less than n) would be

$$Rv^n \left(g - j_{(p)}\right) s^{(p)}_{\overline{m}|}$$

$$= Rv^n g\, s^{(p)}_{\overline{m}|} - Rv^n j_{(p)} s^{(p)}_{\overline{m}|}$$

$$= Rv^{n-m} g\, a^{(p)}_{\overline{m}|} - Rv^n \left\{ (1+i)^m - 1 \right\}$$

$$= Rg\left(a^{(p)}_{\overline{n}|} - a^{(p)}_{\overline{n-m}|}\right) - \left(Rv^{n-m} - Rv^n\right)$$

$$= \left(Rv^n + Rg\, a^{(p)}_{\overline{n}|}\right) - \left(Rv^{n-m} + Rg\, a^{(p)}_{\overline{n-m}|}\right)$$

The sinking fund accumulations at the end of m years thus represent the difference between the value according to a re-valuation at the original rate of interest and the original purchase price.

The above demonstrations hold good irrespective of whether the security was purchased at a premium or at a discount, and whether, therefore, the sinking fund is a true one or a negative one.

(*b*) To show that the values of ${}_tV_{\overline{n}|}$ according to the retrospective and prospective methods are identical if the rate of interest employed throughout the calculations is the same as that at which the premiums were calculated.

$$v^{n-t} - P_{\overline{n}|}\, \ddot{a}_{\overline{n-t}|}$$

$$= v^{n-t} - \frac{a_{\overline{n-t}|}}{s_{\overline{n}|}} \text{ since } \ddot{a}_{\overline{n-t}|} = (1+i)\, a_{\overline{n-t}|} \text{ and } P_{\overline{n}|} = \frac{1}{(1+i)\, s_{\overline{n}|}}$$

$$= v^{n-t} - \frac{1 - v^{n-t}}{(1+i)^n - 1}$$

$$= \frac{(1+i)^n \times v^{n-t} - v^{n-t} - 1 + v^{n-t}}{(1+i)^n - 1}$$

$$= \frac{(1+i)^t - 1}{(1+i)^n - 1}$$

$$= \frac{i s_{\overline{t}|}}{i s_{\overline{n}|}}$$

$$= \frac{(1+i)\, s_{\overline{t}|}}{(1+i)\, s_{\overline{n}|}} \text{ on multiplying numerator and denominator by } \frac{(1+i)}{i}$$

$$= P_{\overline{n}|}\left(s_{\overline{t+1}|} - 1\right) \text{ since } \frac{1}{(1+i)\, s_{\overline{n}|}} = P_{\overline{n}|}$$

$$\text{and } (1+i)\, s_{\overline{t}|} = s_{\overline{t+1}|} - 1.$$

COMPOUND INTEREST

TABLES

1 per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·01000	·99010	1·0000	·9901	1
2	1·02010	·98030	2·0100	1·9704	2
3	1·03030	·97059	3·0301	2·9410	3
4	1·04060	·96098	4·0604	3·9020	4
5	1·05101	·95147	5·1010	4·8534	5
6	1·06152	·94205	6·1520	5·7955	6
7	1·07214	·93272	7·2135	6·7282	7
8	1·08286	·92348	8·2857	7·6517	8
9	1·09369	·91434	9·3685	8·5660	9
10	1·10462	·90529	10·4622	9·4713	10
11	1·11567	·89632	11·5668	10·3676	11
12	1·12683	·88745	12·6825	11·2551	12
13	1·13809	·87866	13·8093	12·1337	13
14	1·14947	·86996	14·9474	13·0037	14
15	1·16097	·86135	16·0969	13·8651	15
16	1·17258	·85282	17·2579	14·7179	16
17	1·18430	·84438	18·4304	15·5623	17
18	1·19615	·83602	19·6147	16·3983	18
19	1·20811	·82774	20·8109	17·2260	19
20	1·22019	·81954	22·0190	18·0456	20
21	1·23239	·81143	23·2392	18·8570	21
22	1·24472	·80340	24·4716	19·6604	22
23	1·25716	·79544	25·7163	20·4558	23
24	1·26973	·78757	26·9735	21·2434	24
25	1·28243	·77977	28·2432	22·0232	25
26	1·29526	·77205	29·5256	22·7952	26
27	1·30821	·76440	30·8209	23·5596	27
28	1·32129	·75684	32·1291	24·3164	28
29	1·33450	·74934	33·4504	25·0658	29
30	1·34785	·74192	34·7849	25·8077	30
31	1·36133	·73458	36·1327	26·5423	31
32	1·37494	·72730	37·4941	27·2696	32
33	1·38869	·72010	38·8690	27·9897	33
34	1·40258	·71297	40·2577	28·7027	34
35	1·41660	·70591	41·6603	29·4086	35
36	1·43077	·69892	43·0769	30·1075	36
37	1·44508	·69200	44·5076	30·7995	37
38	1·45953	·68515	45·9527	31·4847	38
39	1·47412	·67837	47·4123	32·1630	39
40	1·48886	·67165	48·8864	32·8347	40
41	1·50375	·66500	50·3752	33·4997	41
42	1·51879	·65842	51·8790	34·1581	42
43	1·53398	·65190	53·3978	34·8100	43
44	1·54932	·64545	54·9318	35·4555	44
45	1·56481	·63906	56·4811	36·0945	45
46	1·58046	·63273	58·0459	36·7272	46
47	1·59626	·62646	59·6263	37·3537	47
48	1·61223	·62026	61·2226	37·9740	48
49	1·62835	·61412	62·8348	38·5881	49
50	1·64463	·60804	64·4632	39·1961	50

$1\frac{1}{2}$ per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\|}$	$a_{\overline{n}\|}$	n
1	1·01500	·98522	1·0000	·9852	1
2	1·03023	·97066	2·0150	1·9559	2
3	1·04568	·95632	3·0452	2·9122	3
4	1·06136	·94218	4·0909	3·8544	4
5	1·07728	·92826	5·1523	4·7826	5
6	1·09344	·91454	6·2296	5·6972	6
7	1·10984	·90103	7·3230	6·5982	7
8	1·12649	·88771	8·4328	7·4859	8
9	1·14339	·87459	9·5593	8·3605	9
10	1·16054	·86167	10·7027	9·2222	10
11	1·17795	·84893	11·8633	10·0711	11
12	1·19562	·83639	13·0412	10·9075	12
13	1·21355	·82403	14·2368	11·7315	13
14	1·23176	·81185	15·4504	12·5434	14
15	1·25023	·79985	16·6821	13·3432	15
16	1·26899	·78803	17·9324	14·1313	16
17	1·28802	·77639	19·2014	14·9076	17
18	1·30734	·76491	20·4894	15·6726	18
19	1·32695	·75361	21·7967	16·4262	19
20	1·34686	·74247	23·1237	17·1686	20
21	1·36706	·73150	24·4705	17·9001	21
22	1·38756	·72069	25·8376	18·6208	22
23	1·40838	·71004	27·2251	19·3309	23
24	1·42950	·69954	28·6335	20·0304	24
25	1·45095	·68921	30·0630	20·7196	25
26	1·47271	·67902	31·5140	21·3986	26
27	1·49480	·66899	32·9867	22·0676	27
28	1·51722	·65910	34·4815	22·7267	28
29	1·53998	·64936	35·9987	23·3761	29
30	1·56308	·63976	37·5387	24·0158	30
31	1·58653	·63031	39·1018	24·6461	31
32	1·61032	·62099	40·6883	25·2671	32
33	1·63448	·61182	42·2986	25·8790	33
34	1·65900	·60277	43·9331	26·4817	34
35	1·68388	·59387	45·5921	27·0756	35
36	1·70914	·58509	47·2760	27·6607	36
37	1·73478	·57644	48·9851	28·2371	37
38	1·76080	·56792	50·7199	28·8051	38
39	1·78721	·55953	52·4807	29·3646	39
40	1·81402	·55126	54·2679	29·9158	40
41	1·84123	·54312	56·0819	30·4590	41
42	1·86885	·53509	57·9231	30·9941	42
43	1·89688	·52718	59·7920	31·5212	43
44	1·92533	·51939	61·6889	32·0406	44
45	1·95421	·51171	63·6142	32·5523	45
46	1·98353	·50415	65·5684	33·0565	46
47	2·01328	·49670	67·5519	33·5532	47
48	2·04348	·48936	69·5652	34·0426	48
49	2·07413	·48213	71·6087	34·5247	49
50	2·10524	·47500	73·6828	34·9997	50

2 per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·02000	·98039	1·0000	·9804	1
2	1·04040	·96117	2·0200	1·9416	2
3	1·06121	·94232	3·0604	2·8839	3
4	1·08243	·92385	4·1216	3·8077	4
5	1·10408	·90573	5·2040	4·7135	5
6	1·12616	·88797	6·3081	5·6014	6
7	1·14869	·87056	7·4343	6·4720	7
8	1·17166	·85349	8·5830	7·3255	8
9	1·19509	·83676	9·7546	8·1622	9
10	1·21899	·82035	10·9497	8·9826	10
11	1·24337	·80426	12·1687	9·7868	11
12	1·26824	·78849	13·4121	10·5753	12
13	1·29361	·77303	14·6803	11·3484	13
14	1·31948	·75788	15·9739	12·1062	14
15	1·34587	·74301	17·2934	12·8493	15
16	1·37279	·72845	18·6393	13·5777	16
17	1·40024	·71416	20·0121	14·2919	17
18	1·42825	·70016	21·4123	14·9920	18
19	1·45681	·68643	22·8406	15·6785	19
20	1·48595	·67297	24·2974	16·3514	20
21	1·51567	·65978	25·7833	17·0112	21
22	1·54598	·64684	27·2990	17·6580	22
23	1·57690	·63416	28·8450	18·2922	23
24	1·60844	·62172	30·4219	18·9139	24
25	1·64061	·60953	32·0303	19·5235	25
26	1·67342	·59758	33·6709	20·1210	26
27	1·70689	·58586	35·3443	20·7069	27
28	1·74102	·57437	37·0512	21·2813	28
29	1·77584	·56311	38·7922	21·8444	29
30	1·81136	·55207	40·5681	22·3965	30
31	1·84759	·54125	42·3794	22·9377	31
32	1·88454	·53063	44·2270	23·4683	32
33	1·92223	·52023	46·1116	23·9886	33
34	1·96068	·51003	48·0338	24·4986	34
35	1·99989	·50003	49·9945	24·9986	35
36	2·03989	·49022	51·9944	25·4888	36
37	2·08068	·48061	54·0343	25·9695	37
38	2·12230	·47119	56·1149	26·4406	38
39	2·16474	·46195	58·2372	26·9026	39
40	2·20804	·45289	60·4020	27·3555	40
41	2·25220	·44401	62·6100	27·7995	41
42	2·29724	·43530	64·8622	28·2348	42
43	2·34319	·42677	67·1595	28·6616	43
44	2·39005	·41840	69·5027	29·0800	44
45	2·43785	·41020	71·8927	29·4902	45
46	2·48661	·40215	74·3306	29·8923	46
47	2·53634	·39427	76·8172	30·2866	47
48	2·58707	·38654	79·3535	30·6731	48
49	2·63881	·37896	81·9406	31·0521	49
50	2·69159	·37153	84·5794	31·4236	50

$2\frac{1}{2}$ per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·02500	·97561	1·0000	·9756	1
2	1·05063	·95181	2·0250	1·9274	2
3	1·07689	·92860	3·0756	2·8560	3
4	1·10381	90595	4·1525	3·7620	4
5	1·13141	·88385	5·2563	4·6458	5
6	1·15969	·86230	6·3877	5·5081	6
7	1·18869	·84127	7·5474	6·3494	7
8	1·21840	·82075	8·7361	7·1701	8
9	1·24886	·80073	9·9545	7·9709	9
10	1·28008	·78120	11·2034	8·7521	10
11	1·31209	·76214	12·4835	9·5142	11
12	1·34489	·74356	13·7956	10·2578	12
13	1·37851	·72542	15·1404	10·9832	13
14	1·41297	·70773	16·5190	11·6909	14
15	1·44830	·69047	17·9319	12·3814	15
16	1·48451	·67363	19·3802	13·0550	16
17	1·52162	·65720	20·8647	13·7122	17
18	1·55966	·64117	22·3863	14·3534	18
19	1·59865	·62553	23·9460	14·9789	19
20	1·63862	·61027	25·5447	15·5892	20
21	1·67958	·59539	27·1833	16·1845	21
22	1·72157	·58086	28·8629	16·7654	22
23	1·76461	·56670	30·5844	17·3321	23
24	1·80873	·55288	32·3490	17·8850	24
25	1·85394	·53939	34·1578	18·4244	25
26	1·90029	·52623	36·0117	18·9506	26
27	1·94780	·51340	37·9120	19·4640	27
28	1·99650	·50088	39·8598	19·9649	28
29	2·04641	·48866	41·8563	20·4535	29
30	2·09757	·47674	43·9027	20·9303	30
31	2·15001	·46511	46·0003	21·3954	31
32	2·20376	·45377	48·1503	21·8492	32
33	2·25885	·44270	50·3540	22·2919	33
34	2·31532	·43191	52·6129	22·7238	34
35	2·37321	·42137	54·9282	23·1452	35
36	2·43254	·41109	57·3014	23·5563	36
37	2·49335	·40107	59·7339	23·9573	37
38	2·55568	·39128	62·2273	24·3486	38
39	2·61957	·38174	64·7830	24·7303	39
40	2·68506	·37243	67·4026	25·1028	40
41	2·75219	·36335	70·0876	25·4661	41
42	2·82100	·35448	72·8398	25·8206	42
43	2·89152	·34584	75·6608	26·1664	43
44	2·96381	·33740	78·5523	26·5038	44
45	3·03790	·32917	81·5161	26·8330	45
46	3·11385	·32115	84·5540	27·1542	46
47	3·19170	·31331	87·6679	27·4675	47
48	3·27149	·30567	90·8596	27·7732	48
49	3·35328	·29822	94·1311	28·0714	49
50	3·43711	·29094	97·4843	28·3623	50

3 per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rvert}$	$a_{\overline{n}\rvert}$	n
1	1·03000	·97087	1·0000	·9709	1
2	1·06090	·94260	2·0300	1·9135	2
3	1·09273	·91514	3·0909	2·8286	3
4	1·12551	·88849	4·1836	3·7171	4
5	1·15927	·86261	5·3091	4·5797	5
6	1·19405	·83748	6·4684	5·4172	6
7	1·22987	·81309	7·6625	6·2303	7
8	1·26677	·78941	8·8923	7·0197	8
9	1·30477	·76642	10·1591	7·7861	9
10	1·34392	·74409	11·4639	8·5302	10
11	1·38423	·72242	12·8078	9·2526	11
12	1·42576	·70138	14·1920	9·9540	12
13	1·46853	·68095	15·6178	10·6350	13
14	1·51259	·66112	17·0863	11·2961	14
15	1·55797	·64186	18·5989	11·9379	15
16	1·60471	·62317	20·1569	12·5611	16
17	1·65285	·60502	21·7616	13·1661	17
18	1·70243	·58739	23·4144	13·7535	18
19	1·75351	·57029	25·1169	14·3238	19
20	1·80611	·55368	26·8704	14·8775	20
21	1·86029	·53755	28·6765	15·4150	21
22	1·91610	·52189	30·5368	15·9369	22
23	1·97359	·50669	32·4529	16·4436	23
24	2·03279	·49193	34·4265	16·9355	24
25	2·09378	·47761	36·4593	17·4131	25
26	2·15659	·46369	38·5530	17·8768	26
27	2·22129	·45019	40·7096	18·3270	27
28	2·28793	43708	42·9309	18·7641	28
29	2·35657	·42435	45·2189	19·1885	29
30	2·42726	·41199	47·5754	19·6004	30
31	2·50008	·39999	50·0027	20·0004	31
32	2·57508	·38834	52·5028	20·3888	32
33	2·65234	·37703	55·0778	20·7658	33
34	2·73191	·36604	57·7302	21·1318	34
35	2·81386	·35538	60·4621	21·4872	35
36	2·89828	·34503	63·2759	21·8323	36
37	2·98523	·33498	66·1742	22·1672	37
38	3·07478	·32523	69·1594	22·4925	38
39	3·16703	·31575	72·2342	22·8082	39
40	3·26204	·30656	75·4013	23·1148	40
41	3·35990	·29763	78·6633	23·4124	41
42	3·46070	·28896	82·0232	23·7014	42
43	3·56452	·28054	85·4839	23·9819	43
44	3·67145	·27237	89·0484	24·2543	44
45	3·78160	·26444	92·7199	24·5187	45
46	3·89504	·25674	96·5015	24·7754	46
47	4·01190	·24926	100·3965	25·0247	47
48	4·13225	·24200	104·4084	25·2667	48
49	4·25622	·23495	108·5406	25·5017	49
50	4·38391	·22811	112·7969	25·7298	50

3½ per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·03500	·96618	1·0000	·9662	1
2	1·07123	·93351	2·0350	1·8997	2
3	1·10872	·90194	3·1062	2·8016	3
4	1·14752	·87144	4·2149	3·6731	4
5	1·18769	·84197	5·3625	4·5151	5
6	1·22926	·81350	6·5502	5·3286	6
7	1·27228	·78599	7·7794	6·1145	7
8	1·31681	·75941	9·0517	6·8740	8
9	1·36290	·73373	10·3685	7·6077	9
10	1·41060	·70892	11·7314	8·3166	10
11	1·45997	·68495	13·1420	9·0016	11
12	1·51107	·66178	14·6020	9·6633	12
13	1·56396	·63940	16·1130	10·3027	13
14	1·61869	·61778	17·6770	10·9205	14
15	1·67535	·59689	19·2957	11·5174	15
16	1·73399	·57671	20·9710	12·0941	16
17	1·79468	·55720	22·7050	12·6513	17
18	1·85749	·53836	24·4997	13·1897	18
19	1·92250	·52016	26·3572	13·7098	19
20	1·98979	·50257	28·2797	14·2124	20
21	2·05943	·48557	30·2695	14·6980	21
22	2·13151	·46915	32·3289	15·1671	22
23	2·20611	·45329	34·4604	15·6204	23
24	2·28333	·43796	36·6665	16·0584	24
25	2·36324	·42315	38·9499	16·4815	25
26	2·44596	·40884	41·3131	16·8904	26
27	2·53157	·39501	43·7591	17·2854	27
28	2·62017	·38165	46·2906	17·6670	28
29	2·71188	·36875	48·9108	18·0358	29
30	2·80679	·35628	51·6227	18·3920	30
31	2·90503	·34423	54·4295	18·7363	31
32	3·00671	·33259	57·3345	19·0689	32
33	3·11194	·32134	60·3412	19·3902	33
34	3·22086	·31048	63·4532	19·7007	34
35	3·33359	·29998	66·6740	20·0007	35
36	3·45027	·28983	70·0076	20·2905	36
37	3·57103	·28003	73·4579	20·5705	37
38	3·69601	·27056	77·0289	20·8411	38
39	3·82537	·26141	80·7249	21·1025	39
40	3·95926	·25257	84·5503	21·3551	40
41	4·09783	·24403	88·5095	21·5991	41
42	4·24126	·23578	92·6074	21·8349	42
43	4·38970	·22781	96·8486	22·0627	43
44	4·54334	·22010	101·2383	22·2828	44
45	4·70236	·21266	105·7817	22·4955	45
46	4·86694	·20547	110·4840	22·7009	46
47	5·03728	·19852	115·3510	22·8994	47
48	5·21359	·19181	120·3883	23·0912	48
49	5·39606	·18532	125·6018	23·2766	49
50	5·58493	·17905	130·9979	23·4556	50

4 per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·04000	·96154	1·0000	·9615	1
2	1·08160	·92456	2·0400	1·8861	2
3	1·12486	·88900	3·1216	2·7751	3
4	1·16986	·85480	4·2465	3·6299	4
5	1·21665	·82193	5·4163	4·4518	5
6	1·26532	·79031	6·6330	5·2421	6
7	1·31593	·75992	7·8983	6·0021	7
8	1·36857	·73069	9·2142	6·7327	8
9	1·42331	·70259	10·5828	7·4353	9
10	1·48024	·67556	12·0061	8·1109	10
11	1·53945	·64958	13·4864	8·7605	11
12	1·60103	·62460	15·0258	9·3851	12
13	1·66507	·60057	16·6268	9·9856	13
14	1·73168	·57748	18·2919	10·5631	14
15	1·80094	·55526	20·0236	11·1184	15
16	1·87298	·53391	21·8245	11·6523	16
17	1·94790	·51337	23·6975	12·1657	17
18	2·02582	·49363	25·6454	12·6593	18
19	2·10685	·47464	27·6712	13·1339	19
20	2·19112	·45639	29·7781	13·5903	20
21	2·27877	·43883	31·9692	14·0292	21
22	2·36992	·42196	34·2480	14·4511	22
23	2·46472	·40573	36·6179	14·8568	23
24	2·56330	·39012	39·0826	15·2470	24
25	2·66584	·37512	41·6459	15·6221	25
26	2·77247	·36069	44·3117	15·9828	26
27	2·88337	·34682	47·0842	16·3296	27
28	2·99870	·33348	49·9676	16·6631	28
29	3·11865	·32065	52·9663	16·9637	29
30	3·24340	·30832	56·0849	17·2920	30
31	3·37313	·29646	59·3283	17·5885	31
32	3·50806	·28506	62·7015	17·8736	32
33	3·64838	·27409	66·2095	18·1476	33
34	3·79432	·26355	69·8579	18·4112	34
35	3·94609	·25342	73·6522	18·6646	35
36	4·10393	·24367	77·5983	18·9083	36
37	4·26809	·23430	81·7022	19·1426	37
38	4·43881	·22529	85·9703	19·3679	38
39	4·61637	·21662	90·4091	19·5845	39
40	4·80102	·20829	95·0255	19·7928	40
41	4·99306	·20028	99·8265	19·9931	41
42	5·19278	·19257	104·8196	20·1856	42
43	5·40050	·18517	110·0124	20·3708	43
44	5·61652	·17805	115·4129	20·5488	44
45	5·84118	·17120	121·0294	20·7200	45
46	6·07482	·16461	126·8706	20·8847	46
47	6·31782	·15828	132·9454	21·0429	47
48	6·57053	·15219	139·2632	21·1951	48
49	6·83335	·14634	145·8337	21·3415	49
50	7·10668	·14071	152·6671	21·4822	50

$4\frac{1}{2}$ per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·04500	·95694	1·0000	·9569	1
2	1·09203	·91573	2·0450	1·8727	2
3	1·14117	·87630	3·1370	2·7490	3
4	1·19252	·83856	4·2782	3·5875	4
5	1·24618	·80245	5·4707	4·3900	5
6	1·30226	·76790	6·7169	5·1579	6
7	1·36086	·73483	8·0192	5·8927	7
8	1·42210	·70319	9·3800	6·5959	8
9	1·48610	·67290	10·8021	7·2688	9
10	1·55297	·64393	12·2882	7·9127	10
11	1·62285	·61620	13·8412	8·5289	11
12	1·69588	·58966	15·4640	9·1186	12
13	1·77220	·56427	17·1599	9·6829	13
14	1·85194	·53997	18·9321	10·2228	14
15	1·93528	·51672	20·7841	10·7395	15
16	2·02237	·49447	22·7193	11·2340	16
17	2·11338	·47318	24·7417	11·7072	17
18	2·20848	·45280	26·8551	12·1600	18
19	2·30786	·43330	29·0636	12·5933	19
20	2·41171	·41464	31·3714	13·0079	20
21	2·52024	·39679	33·7831	13·4047	21
22	2·63365	·37970	36·3034	13·7844	22
23	2·75217	·36335	38·9370	14·1478	23
24	2·87601	·34770	41·6892	14·4955	24
25	3·00543	·33273	44·5652	14·8282	25
26	3·14068	·31840	47·5706	15·1466	26
27	3·28201	·30469	50·7113	15·4513	27
28	3·42970	·29157	53·9933	15·7429	28
29	3·58404	·27902	57·4230	16·0219	29
30	3·74532	·26700	61·0071	16·2889	30
31	3·91386	·25550	64·7524	16·5444	31
32	4·08998	·24450	68·6662	16·7889	32
33	4·27403	·23397	72·7562	17·0229	33
34	4·46636	·22390	77·0303	17·2468	34
35	4·66735	·21425	81·4966	17·4610	35
36	4·87738	·20503	86·1640	17·6660	36
37	5·09686	·19620	91·0413	17·8622	37
38	5·32622	·18775	96·1382	18·0500	38
39	5·56590	·17967	101·4644	18·2297	39
40	5·81636	·17193	107·0303	18·4016	40
41	6·07810	·16453	112·8467	18·5661	41
42	6·35161	·15744	118·9248	18·7235	42
43	6·63744	·15066	125·2764	18·8742	43
44	6·93612	·14417	131·9138	19·0184	44
45	7·24825	·13796	138·8500	19·1563	45
46	7·57442	·13202	146·0982	19·2884	46
47	7·91527	·12634	153·6726	19·4147	47
48	8·27145	·12090	161·5879	19·5356	48
49	8·64367	·11569	169·8594	19·6513	49
50	9·03264	·11071	178·5030	19·7620	50

5 per cent

n	$(1+i)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	n
1	1·05000	·95238	1·0000	·9524	1
2	1·10250	·90703	2·0500	1·8594	2
3	1·15763	·86384	3·1525	2·7232	3
4	1·21551	·82270	4·3101	3·5460	4
5	1·27628	·78353	5·5256	4·3295	5
6	1·34010	·74622	6·8019	5·0757	6
7	1·40710	·71068	8·1420	5·7864	7
8	1·47746	·67684	9·5491	6·4632	8
9	1·55133	·64461	11·0266	7·1078	9
10	1·62889	·61391	12·5779	7·7217	10
11	1·71034	·58468	14·2068	8·3064	11
12	1·79586	·55684	15·9171	8·8633	12
13	1·88565	·53032	17·7130	9·3936	13
14	1·97993	·50507	19·5986	9·8986	14
15	2·07893	·48102	21·5786	10·3797	15
16	2·18287	·45811	23·6575	10·8378	16
17	2·29202	·43630	25·8404	11·2741	17
18	2·40662	·41552	28·1324	11·6896	18
19	2·52695	·39573	30·5390	12·0853	19
20	2·65330	·37689	33·0660	12·4622	20
21	2·78596	·35894	35·7193	12·8212	21
22	2·92526	·34185	38·5052	13·1630	22
23	3·07152	·32557	41·4305	13·4886	23
24	3·22510	·31007	44·5020	13·7986	24
25	3·38635	·29530	47·7271	14·0939	25
26	3·55567	·28124	51·1135	14·3752	26
27	3·73346	·26785	54·6691	14·6430	27
28	3·92013	·25509	58·4026	14·8981	28
29	4·11614	·24295	62·3227	15·1411	29
30	4·32194	·23138	66·4388	15·3725	30
31	4·53804	·22036	70·7608	15·5928	31
32	4·76494	·20987	75·2988	15·8027	32
33	5·00319	·19987	80·0638	16·0025	33
34	5·25335	·19035	85·0670	16·1929	34
35	5·51602	·18129	90·3203	16·3742	35
36	5·79182	·17266	95·8363	16·5469	36
37	6·08141	·16444	101·6281	16·7113	37
38	6·38548	·15661	107·7095	16·8679	38
39	6·70475	·14915	114·0950	17·0170	39
40	7·03999	·14205	120·7998	17·1591	40
41	7·39199	·13528	127·8398	17·2944	41
42	7·76159	·12884	135·2318	17·4232	42
43	8·14967	·12270	142·9933	17·5459	43
44	8·55715	·11686	151·1430	17·6628	44
45	8·98501	·11130	159·7002	17·7741	45
46	9·43426	·10600	168·6852	17·8801	46
47	9·90597	·10095	178·1194	17·9810	47
48	10·40127	·09614	188·0254	18·0772	48
49	10·92133	·09156	198·4267	18·1687	49
50	11·46740	·08720	209·3480	18·2559	50

Table of $(1+i)^{\frac{1}{p}}$

·01	1·00499	1·00249	1·00083
·015	1·00747	1·00373	1·00124
·02	1·00995	1·00496	1·00165
·025	1·01242	1·00619	1·00206
·03	1·01489	1·00742	1·00247
·035	1·01735	1·00864	1·00287
·04	1·01980	1·00985	1·00327
·045	1·02225	1·01107	1·00367
·05	1·02470	1·01227	1·00407

Table of $v^{\frac{1}{p}}$

i	p		
	2	4	12
·01	·99504	·99752	·99917
·015	·99258	·99628	·99876
·02	·99015	·99506	·99835
·025	·98773	·99385	·99794
·03	·98533	·99264	·99754
·035	·98295	·99144	·99714
·04	·98058	·99024	·99674
·045	·97823	·98906	·99634
·05	·97590	·98788	·99594

Table of $\dfrac{i}{p\left\{(1+i)^{\frac{1}{p}}-1\right\}}$ or $\dfrac{i}{j_{(p)}}$

	2	4	12
·01	1·00249	1·00374	1·00458
·015	1·00374	1·00561	1·00686
·02	1·00498	1·00747	1·00913
·025	1·00621	1·00933	1·01141
·03	1·00744	1·01118	1·01368
·035	1·00867	1·01303	1·01594
·04	1·00990	1·01488	1·01820
·045	1·01113	1·01672	1·02046
·05	1·01235	1·01856	1·02271

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(8) (a) 4·909%; (b) 4·841%.
(9) £224 13s. 0d.
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(12) $\frac{£12}{(1 \cdot 05)^{10} - (1 \cdot 035)^{10}} = £55.$ $£55\,(1 \cdot 05)^{10} = £89$ 11s. 9

EXERCISE II

(1) £466 0s. 10d.; £1,221 2s. 2d.
(2) £816 14s. 3d; £7,343 11s. 3d.
(3) £761 1s. 3d.; £1,539 3s. 1d.
(4) £358 17s. 11d.; £1,979 13s. 9d.
(5) £25 ÷ ·06 = £416 13s. 4d.
(6) $£150 \times a_{\overline{25}|} - £25 \times a_{\overline{7}|} = £2{,}193$ 5s. 0d approx.

$s_{\overline{n}|} = \frac{1}{\cdot 05}(3 \cdot 071524 - 1) = 41 \cdot 4305.$

(8) $£50 \times {}_{7}a_{\overline{5}|} = £50\,(a_{\overline{12}|} - a_{\overline{7}|}) = £154.$

10) $\ddot{a}_{\overline{n}|} = (1 + i) \times a_{\overline{n}|}$. Hence $(1 + i) = \frac{[illegible]}{29 \cdot 0800} = 1 \cdot 02$ and $i = \cdot 02$.

EXERCISE III

(1) $£25 \cdot 175 \times \frac{i}{j_{(2)}} \times a_{\overline{20}|} = £345$ 10s. 6d.;

$£25 \cdot 175 \times \frac{i}{j_{(2)}} \times s_{\overline{20}|} = £757$ 1s. 9d.

(2) $£12 \cdot 4167 \times \frac{i}{j_{(2)}} \times a_{\overline{10}|} \times (1 + i)^{\frac{1}{2}} = £99$ 9s. 2d.;

$£12 \cdot 4167 \times \frac{i}{j_{(2)}} \times s_{\overline{10}|} \times (1 + i)^{\frac{1}{2}} = £162$ 0s. 3d.

(3) (a) $£35 \cdot 925 \times \frac{i}{j_{(4)}} \times (a_{\overline{15}|} - a_{\overline{5}|}) = £254$ 16s. 9d.

(b) Previous result $\times (1{\cdot}035)^{\frac{1}{4}}$ = £257 0s. 9d.

(4) (a) £40 $\times \dfrac{i}{j_{(2)}} \times \dfrac{1}{i} \times (1+i)^{\frac{1}{4}}$ = £819 16s. 4d.;

(b) £20 $\times \dfrac{1}{i} \times (1+i)^{\frac{1}{3}}$ @ $2\frac{1}{2}\%$ = £809 18s. 9d.

(5) £22·75 $\times a_{\overline{40}|}$ @ 2% = £622 6s. 9d.;

£22·75 $\times s_{\overline{40}|}$ @ 2% = £1,374 2s. 11d.

(6) £24 $\times a^{(2)}_{\overline{20}|}$ @ $2\frac{1}{2}\%$ = £376 9s. 4d.;

£24 $\times s^{(2)}_{\overline{20}|}$ @ $2\frac{1}{2}\%$ = £616 17s. 7d.

Amount = $(1{\cdot}015)^{32}$ × Present value = £100 18s. 11d.

(8) £85·3833 $\times \dfrac{i}{j_{(4)}}$ @ 5% = £86 19s. 4d.

EXERCISE IV

(1) (a) £102 v^5 + £4 $a^{(2)}_{\overline{5}|}$ @ 5% = £97 9s. 1d.;

(b) £103 v^{16} + £2·5 $a_{\overline{16}|}$ @ 3% = £95 11s. 10d.;

(c) £106 v^{26} + £2·5 $a^{(2)}_{\overline{26}|}$ @ 3% = £94 3s. 6d.

(2) £98 5s 6d.; £99 2s. 10d.; £100 1s. 0d.; £101 0s. 0d.; £102 0s. 0d.

(3) $\dfrac{£2{\cdot}067}{s_{\overline{10}|}}$ @ 2%, or £2·25 − £·02 (103·067) = 3s. 9d. approx.

(4) £102 v^{10} + £4·5 $a_{\overline{10}|}$ @ 5% = £97·367;

£97·367 v^5 + £4·5 $a_{\overline{5}|}$ @ 5% = £95·772 = original purchase price,

viz. £102 v^{15} + £4·5 $a_{\overline{15}|}$ @ 5%.

104 +

(6) Let x = rate of interest per half-year,

$x \times 107 = 2{\cdot}75 - \dfrac{3}{s_{\overline{20}|}}$ @ 2% = 2·627

Whence x = ·02455, and nominal rate of interest convertible half-yearly = 4·91% per annum.

(7) Let R = redemption price

$$R\,v^{15} + 4{\cdot}5\,a^{(2)}_{\overline{15}|} = 98 \text{ @ } 5\%$$

$$R = 98\,(1{\cdot}05)^{15} - 4{\cdot}5 \times \frac{i}{j_{(2)}} \times s_{\overline{15}|} \text{ @ } 5\% = £105{\cdot}4 \text{ approx.}$$

(8) (1 + ·3836 × ·06) (102 v^{30} + 2 $a_{\overline{30}|}$) @ 3% = £83 1s. 10d.

EXERCISE V

(1) £28 17s. 5d.

(2)

Year.	Principal outstanding at beginning of year.	Interest included in payment for year.	Principal included in payment for year.
	£ s. d.	£ s. d.	£ s. d.
1	125 – –	6 5 –	22 12 5
2	102 7 7	5 2 4	23 15 1
3	78 12 6	3 18 7	24 18 10
4	53 13 8	2 13 8	26 3 9
5	27 9 11	1 7 6	27 9 11

(3) £46 3s.

Half-year No.	Principal outstanding at beginning of half-year.	Interest included in payment for half-year.	Principal included in payment for half-year.
	£ s. d.	£ s. d.	£ s. d.
1	250 – –	7 10 –	38 13 –
2	211 7 –	6 6 10	39 16 2
3	171 10 10	5 2 11	41 – 1
4	130 10 9	3 18 4	42 4 8
5	88 6 1	2 13 –	43 10 –
6	44 16 1	1 6 11	44 16 1

(4) £29 16s. 8d.

Year.	Principal outstanding at beginning of year.	Interest included in payment for year.	Principal included in payment for year.
	£ s. d.	£ s. d.	£ s. d.
1	150 – –	7 10 –	22 6 8
2	127 13 4	6 9 11	23 6 9
3	104 6 7	5 8 11	24 7 9
4	79 18 10	4 7 –	25 9 8
5	54 9 2	3 4 1	26 12 7
6	27 16 7	2 – 1	27 16 7

(5) £95 10s. 10d.
(6) (*a*) £2 16s. 7d.; (*b*) £183 12s. 7d.
(7) £4 14s. 0d.

EXERCISE VI

(1) £47 15s. 3d.
(2) Annual £16 10s. 10d.; Half-yearly £8 6s. 10d.; Quarterly £4 3s. 9d.
(3) £55 7s. 5d.; £91 16s. 3d.
(4) £85 5s. 5d.
(5) £138 4s. 0d. approx.
(6) £1 11s. 2d.
(7) £1 18s. 10d.

EXERCISE VII

(1) 5·478% per annum convertible half-yearly.
(2) 3·120% per annum.
(3) 4·760% per annum.
(4) 5·800% per annum.
(5) 5·380% per annum, convertible half-yearly.
(6) 4·869% per annum, convertible half-yearly.
(7) 5·212% per annum.

EXERCISE VIII

£365 17s. 3d.
£106 11s. 1d.
£35 5s. 8d.
4·641%.
4·183%.
4% per annum, almost exactly.
6·061%.

EXERCISE IX

(1) ·0900 ; 25·518 ; 17·380 ; 174·445.
(2) ·17529 ; 19·1987.
(3) (a)

$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
·9677	1·0000
1·9043	2·0333
2·8106	3·1011
3·6877	4·2045
4·5365	5·3446

$\frac{1}{i}(5 - a_{\overline{5}\rceil})$ calculated at $3\frac{1}{3}\%$

$\frac{1}{i}(s_{\overline{6}\rceil} - 6)$ calculated at $3\frac{1}{3}\%$

(4)

n	v^n	$(1 + i)^n$
1	·9766	1·0240
2	·9537	1·0486
3	·9313	1·0737
4	·9095	1·0995
5	·8882	1·1259

(5) $\frac{v^{25} - v^{40}}{i}$; $\frac{(1 + i)^{11} - (1 +}{i}$

$\frac{1}{i}\left\{15 - (a_{\overline{40}\rceil} - a_{\overline{25}\rceil})\right\}$; $\frac{1}{i}(s_{\overline{41}\rceil} -$

ANSWERS TO MISCELLANEOUS EXERCISES

(1) Multiply half-yearly premium by ($s_{\overline{61}|} - 1$), i.e. by $\{s_{\overline{40}|} \times (1 + i)^{21} + s_{\overline{21}|} - 1\}$ calculated at 2%.

(2) Yearly payment £28·201

Year.	Principal outstanding at beginning of year.	Principal included in yearly payment	Interest included in yearly payment.
	£	£	£
1	100·000	5·000	23·201
2	76·799	3·840	24·361
3	52·438	2·621	25·580
4	26·858	1·343	26·858

(3) ·0685.
(4) £77·06 or £77 1s. 2d. approx.
(5) 6·71% per annum, payable half-yearly (derived from approximate formula without using compound interest tables).
(6) £71 10s. 1d.
(7) £3,040.
(8) £45 10s. 1d.
(9) £332 2s. 3d.
(10) Annual payment £93 10s. 7d.

Year.	Principal outstanding at beginning of year.	Principal included in yearly payment	Interest included in yearly payment.
	£ s. d.	£ s. d.	£ s. d.
1	250 – –	15 – –	78 10 7
2	171 9 5	10 5 10	83 4 9
3	88 4 8	5 5 11	88 4 8

(11) Valuation formula £105 × v^8 + £2·5 × $a_{\overline{8}|}$ @ $2\frac{1}{4}$%.
Effective yield £4 11s. 0d.% per annum.

(12) $3\frac{1}{8}$% per annum, almost exactly.

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